GPS-UTM Module 5: What’s the Angle?

**Topics Covered:** Right angle trigonometry formulas, trig functions and inverse trig functions on the calculator, Law of Cosines

**Required Background Material:** GPS-UTM Module 3, familiarity with similar triangles, use of a protractor

**Introduction**

The triangle is probably used more frequently than any other geometrical shape. Why? Because a triangle is the only rigid shape that can be made with pinned joints - so there is only one set of angles that will go with a given set of sides. That is not true for geometrical figures such as a rectangle or a pentagon.

Look closely at the following picture of the pedestrian bridge between the Science Building and the Biotechnology Building on the Marshall University Huntington campus, and notice how the architects have used triangles to strengthen the structure.

![Pedestrian Bridge](image)

Two triangles are *similar* if the angles of one triangle are congruent to the angles of the other triangle. One triangle may be large, and one may be small; but they have the same shape because their angles are the same. An example occurs when you form the
altitude to the hypotenuse in a right triangle; it forms two new triangles that are similar to the original triangle. This works because both of the new triangles have one acute angle in common with the original right triangle. And, since the measure of the angles in a triangle must add to $180^\circ$; if two right triangles have one pair of congruent acute angles, then they have two pairs of congruent acute angles.

You probably also know that corresponding sides of similar triangles have the same ratio. This is the basis for a branch of mathematics called trigonometry. In the table below are the three most important definitions of trigonometry, based on the following diagram.

<table>
<thead>
<tr>
<th>Trigonometric Ratio</th>
<th>Abbreviation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine</td>
<td>sin</td>
<td>$\sin A = \frac{\text{opposite leg}}{\text{hypotenuse}}$</td>
</tr>
<tr>
<td>cosine</td>
<td>cos</td>
<td>$\cos A = \frac{\text{adjacent leg}}{\text{hypotenuse}}$</td>
</tr>
<tr>
<td>tangent</td>
<td>tan</td>
<td>$\tan A = \frac{\text{opposite leg}}{\text{adjacent leg}}$</td>
</tr>
</tbody>
</table>

The values of the sine, cosine, and tangent are constant for each acute angle because all right triangles with that angle will be similar. So, if we measure $\angle A$ and look up its trig functions, in a trig table or calculator, we only need to measure one side of the triangle in order to figure out how long the other sides are.

Working the other way, if we know the three sides of a triangle, the trig functions of all the angles are known. Using a calculator, we can use inverse trig functions ($\sin^{-1}$ instead of $\sin$, for example) to find the size of each angle. And, if the triangle we are working with is a right triangle, we only need to know the length of two sides. That’s because we can use the Pythagorean Theorem to get the third side.
Look at the sample right triangle below. Notice that side opposite $\angle A$ is called $a$, the side opposite $\angle B$ is called $b$, and the side opposite $\angle C$ is called $c$. This is standard labeling for a triangle. The diagram does not give us the value of $b$, but by using the Pythagorean Theorem we can easily see that it must be equal to 4.

Using the definition of the trig functions, we can compute their values for any of the three angles. For $\angle A$ we see that $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$ and $\tan A = \frac{3}{4}$. To find the value of $\angle A$, set the mode on your calculator to degrees, type in $\tan^{-1} \left( \frac{2\text{nd}}{\tan} \right)$ and then enter the value .75. When you push Enter, you will see that $\angle A = 36.87^\circ$.

**Problem 1**

Find the sine, cosine, and tangent of angle $B$ in the following triangle. Leave your answer in decimal form.

$$\sin B = \underline{\hphantom{10}} \quad \cos B = \underline{\hphantom{10}} \quad \tan B = \underline{\hphantom{10}}$$

What is the measure of $\angle B$?
**Problem 2**

The sine and cosine are always less than or equal to 1. Why?

Can the tangent be greater than one? Why, or why not?

**Problem 3**

Consider the data you found in Module 3 for your football field. What are the measures of the angles in the blue triangle drawn below? Write the answers in the boxes provided.

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**The Law of Cosines**

Unfortunately, not all triangles are right triangles. To find the angles for an arbitrary triangle, we need to know more than the ratio of two sides. As stated before, we need to know the length of all three sides. We can’t always use the Pythagorean Theorem, but the Law of Cosines works every time.

Given an arbitrary triangle ABC, with standard lettering, the *Law of Cosines* is:
We will illustrate this with two examples.

Example 1: Find the value of \( b \) in the triangle above.

Solution: The Law of Cosines is arranged to make it easy to find side \( c \), but we want to find side \( b \); so we need to re-label the sides in the equation. This gives

\[
b^2 = a^2 + c^2 - 2ac \cos B
\]

\[
b^2 = 32^2 + 48^2 - 2(32)(48) \cos 125.2^\circ
\]

\[
b^2 \approx 5098.8
\]

\[b \approx 71.\]

Example 2: Find the measure of \( \angle A \) in the same triangle.

Solution: This time we will re-label the Law of Cosines in order to use angle \( A \). This gives

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

\[
32^2 = 71^2 + 48^2 - 2(71)(48) \cos A
\]

\[
1024 = 5041 + 2304 - 6816 \cos A
\]

\[
-6321 = -6816 \cos A
\]

\[
\cos A \approx 0.9273768
\]

\[A \approx 22.0^\circ\]
Problem 4

Given \( \triangle ABC \), where \( A = 30^\circ \), \( b = 12 \), and \( c = 24 \); find the length of side \( a \).

\[
a = \phantom{0000}\
\]

Problem 5

For the triangle shown above, find the measure of \( \angle T \).

\[
m\angle T = \phantom{000}\
\]

Problem 6 (Hide and Seek)

Now you are ready to apply your skills to a simulated crisis situation. Go outside with two other students. All three of you will need to take cell phones and GPS-UTM devices set in WAAS mode. Two students, the “seekers” will also need protractors, calculators, paper, and pencils.

First the “injured person” (Cindy in the diagram on the next page) will run away and hide. She will then call for help on her cell phone and give the seekers her GPS coordinates. Enter these in the table on the next page.
The two seekers will then establish a base line between them. (See the line between Bill and Amy in the drawing below.) This line can be a straight section of a road or path. Alternately, they could carry compasses and one individual could stand straight north of the other. After the base line is established, the seekers will take up positions and call each other to exchange GPS coordinates. Add these to the table above.

Now draw a diagram similar to this and copy it in the space provided on the next page. Each seeker will use the distance formula to compute the lengths of the sides of the triangle and will write these distances along the appropriate sides in the drawing. Then, using the Law of Cosines, the two seekers should find the measure of the angle in their corner of the triangle. In the sample diagram, the Greek letters $\alpha$ (alpha) and $\beta$ (beta) are used to represent the measure of these angles in degrees. In the sample drawing, Amy will calculate $\alpha$, and Bill will calculate $\beta$.

Finally, each seeker will use his or her protractor to determine which direction to walk (in relation to the base line) in order to find the “injured” person. This process is called triangulation.
Put your diagram here.

Did you find your “injured” person? What difficulties did you have?

**Problem 7 (Opt.)**

Suppose in our example, that Bill and Amy knew the measure of $\alpha$ and $\beta$, but they did not know the distance each was from Cindy. If they knew how far they were from each other, would that be enough to find Cindy? Why or why not?

If you have taken a course in geometry, what congruence theorems have been used in this module?