Ethan Frazier, Chris Kirk, Ryan Jarrell

Calculus III Project, Fall 2017

Abstract: Our problem statement is, "What does the trajectory of a satellite become when its rocket engine is turned on?" This problem statement suggests there are a number of parameters left to the student's discretion. These include questions such as, "Is the initial orbit circular or elliptical?"; "Do the rockets fire once or multiple times?"; "Does the rocket fire so as to move the satellite out of the initial plane of orbit?"; "Should we include only the gravitational field of the Earth, or should we include that of the moon or other bodies?" We decided on the following simplifying assumptions:

(i) The initial orbit is circular
(ii) The rocket may fire more than once
(iii) The rocket may fire so as to preserve the plane of our initial orbit
(iv) We will consider only the gravitational force of the Earth, around which the satellite orbits
(v) We will consider the mass of the satellite negligible relative to that of the Earth

Once the problem had been solved with these initial simplifying assumptions, we agreed that we could, time permitting, attempt to generalize our solution (e.g., by permitting the initial orbit to be an ellipse).

Our hypothesis is that a single brief firing of the rocket will perturb the circular orbit, creating an elliptical one. We also discussed the possibility that periodic firings could be used to achieve a new, approximately circular orbit.

First solution attempt: Finding a polygonal approximation to the trajectory seemed an appropriate place to start. Making the assumption that (iii) above holds, we began our work in $\mathbb{R}^2$. Placing the Earth at the origin, we used the vector equation given in class, $F(\vec{x}) = GM\frac{-\vec{x}}{\|\vec{x}\|^3}$ (ignoring m, the negligible mass of the satellite), for the gravitational pull of the Earth toward the origin. Assuming a circular orbit simplifies the problem by allowing for a constant speed, s, where

$$s = \sqrt{\frac{G(M+m)}{r}}, \text{which becomes simply } \sqrt{\frac{GM}{r}},$$

where r is the radius of orbit. Parametrizing the circular curve of our initial orbit so that the magnitude of the tangent vector is $\sqrt{\frac{GM}{r}}$ at each time t, we have the parametrization

$$P(t) = \langle r\cos\alpha t\,, r\sin\alpha t\rangle, \text{where } \alpha = \sqrt{\frac{GM}{r^3}}.$$

Let $\vec{r}_i$ be the vector representing the force resulting from firing the rocket the ith time; it seemed natural to let $\|\vec{r}_1\| = \|\vec{P}'(t_0)\| = s$, where t_0 is the time of the first firing. Let θ be the angle that $\vec{r}_1$ makes with the tangent vector $P'(t_0)$; initial assumptions were that $\theta = 0$ or $\theta = \frac{\pi}{2}$. Using $F(\vec{x}), P(t)$, and our assumptions about the direction and magnitude of $\vec{r}_1$, we began calculating a polygonal path that would approximate the actual trajectory of the satellite. Since the speed s along the initial orbit is measured in meters per second, it made sense to take the time intervals of length equal to one second: If $t_0, t_1, \cdots, t_n$ are our times, then

$$\Delta t = t_{k+1} - t_k = 1 \text{ second}, \qquad \text{for } k = 0, 1, \cdots, n.$$

Our orbit is circular until time t_0, when the rocket is fired for one second. Assuming $\theta = \frac{\pi}{2}$, our position one second later, at time t_1, is given by the vector sum

$$H(t_1) = P'(t_0) + \vec{r}_1.$$

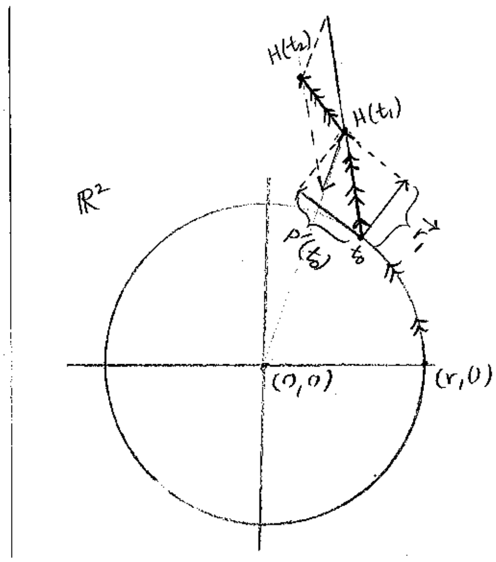

Figure 1

Our satellite would continue traveling along this vector, $P'(t_0) + \vec{r}_1$, were it not for the gravitational influence of the Earth. Factoring in this influence of Earth's gravity, our position one second later, at time t_2, is given by the vector sum

$$H(t_2) = F\big(H(t_1)\big) + H(t_1).$$

Continuing recursively in this manner, we thought, would give us a polygonal approximation to the actual trajectory resulting from firing the rockets. Moreover, letting $\Delta t \longrightarrow 0$ would allow us to achieve any degree of accuracy we pleased (given the assumptions listed above). However, repeated attempts (many, many attempts over several weeks, as a group and as individuals) to actually carry out this computation resulted in nonsense answers (such as the path quickly barreling into the origin).

This still seems like the most sensible plan, though we were not able to identify what was wrong in our computations; and the general approach seemed sound. After giving it considerable thought, we scrapped the idea and took another approach. However, there was one helpful insight that resulted from this: We realized that if the trajectory of the satellite can be approximated by a sum of vectors, and if all these vectors are coplanar, it follows that after the rocket is fired, the new trajectory (regardless of whether it is a circle, ellipse, parabola, or hyperbola), will naturally be a space curve lying in a plane. Only firing the rocket again (or the influence of another celestial body, unaccounted for in our model) can again move the satellite out of the newly-established plane.

Second solution attempt: Frustrated with trying to approximate the trajectory in the method described above, we looked for another way to approach the problem. If the initial and final orbits were circles whose radii were known, could we use knowledge of the Hohmann transfer to produce an intermediary elliptical orbit that would take us from one circular orbit to the other? Attempting to answer this question gave us our second approach to the problem. With the initial and final velocities of the two circular orbits known, we calculated the period of orbit of the Hohmann transfer. All that remained was to discover the angle and magnitude of the vector (representing the force of the rocket) that would allow us to achieve the Hohmann transfer orbit (Figure 2). We discussed how this might be done, and eventually found ourselves facing the original problem in another guise: Now, instead of trying to determine the resulting trajectory given a firing of the rocket, the question had become how to fire the rocket so as to produce a given elliptical trajectory.

We have not found a solution to our question, though we have gained a better understanding of the problem, and an appreciation of its difficulty. As we thought critically about this problem, we found that although answers were in short supply, interesting questions abounded: If we agree to fire the rocket only once at time t_0, and if $\vec{r}$ is at right angles to $P'(t_0)$, what must the magnitude of $\vec{r}$ be to achieve a new trajectory that is (i) elliptical, (ii) parabolic, or (iii) hyperbolic? *Can* each of these new trajectories be achieved simply by varying the magnitude of $\vec{r}$? How does the problem change if the initial orbit is not circular but elliptical? Given an initial circular orbit, if the rocket is fired to produce a new orbit (circular, elliptical, or otherwise), how must the rocket be fired again so as to regain the original circular orbit? These questions and others like them were brainstormed in an attempt to discover an interesting, but more tractable, problem.

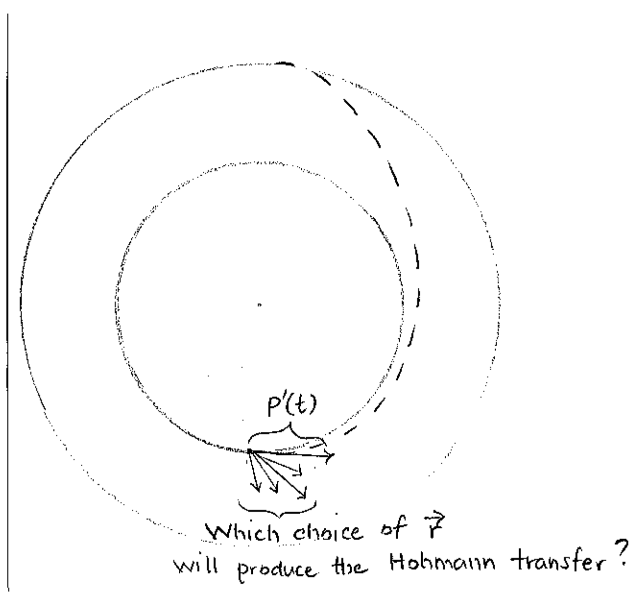

Figure 2