Bailey Bowling
Calculus I Project
MTH 229-105

Bouncing a Ball Down Stairs

For my final project in calculus, I chose the scenario of a ball bouncing down stairs. This question was worded as "How do I throw a ball down a staircase so that it bounces off each step?" I interpreted this problem as asking how to get a ball to bounce down every step in a stair case. At first, I attempted to use physics to solve this problem. My first theory was to use the coefficient of restitution to explain how to optimize your results of such an experiment. The coefficient of restitution is essentially the difference between the maximum heights of two consecutive ball bounces. This loss of height essentially represents the energy lost when a ball

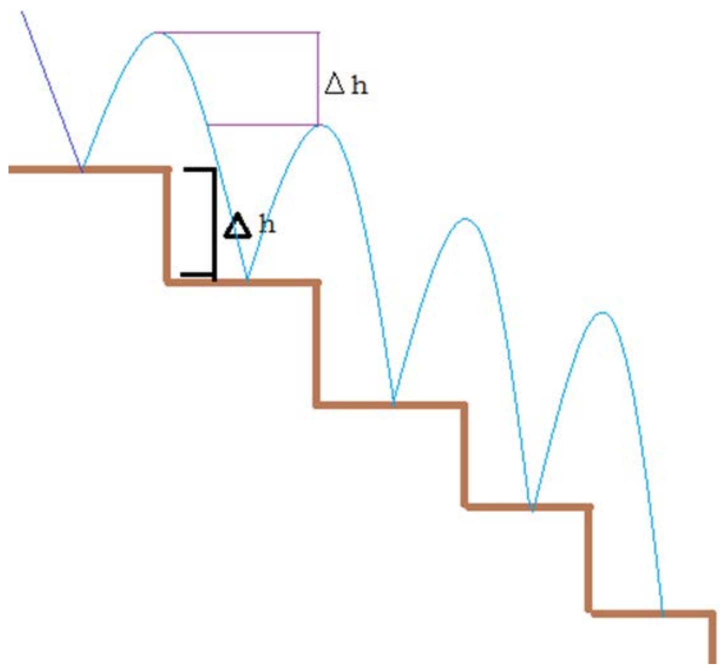

Figure 1

bounces. My original idea was that if the difference between bounce peaks equaled the rise of the step it was bouncing down, the ball should keep bouncing down the stairs without a significant loss of energy. (*Fig. 1)* I soon realized that this physics perspective would be relatively difficult to mathematically represent using only Calculus I principles. However, when correcting this mistake with my later findings I maintained that there would be a relatively small loss of energy among the bounces due to the increase in potential energy from the decrease in stair height.

There were several pieces of information that I assumed when conducting my analysis of this problem. I assumed that there would be no spin put on the ball when throwing it, and that there was not a third dimension to consider in this problem. I also assumed that the ball would be thrown from a height of four feet above the first stair. I decided that the total rise of

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the stairs would be 10 feet and that the total run would be 11 feet 8 inches. The rise of one stair would be 8 inches, and the run would be 10 inches.

To develop the path of the bouncing ball I used the standard form for a concave down quadratic function, $y=-a(x-h)^2-k$, which is derived from the method of "completing the square" (Calculus by Rogawski, Third edition, Section 1.2). The variable "a" is a constant that determines the concavity of the parabola. I determined "a" would equal sixteen because the derivative of my position function would be $y'=-2a(x-h)$, and the second derivative of my function would be $y''= -2a$. I knew that the second derivative is the same thing as acceleration, and I knew my acceleration should be the acceleration due to earth's gravity, 32 feet per second per second (Calculus by Rogawski, Third Edition, Section 3.4)This meant that the value of "a" would need to equal 16. I then determined that "h" and "k" would change for each parabola. These values can be calculated by using the known x and y values from the point where the previous parabola touches the step. For each parabola, h increases by one, because the vertex of each parabola is one second apart. Using this information, you can determine what the k value is by setting the new equation equal to the previous one. Altogether, the parabola equations complete a piecewise function where,

When $0 \le x \le 0.5$, $y= -16(x-0)^2+14$

$0.5 \le x \le 1.54$, $y= -16(x-1)^2+14$

$1.54 \le x \le 2.503$, $y= -16(x-2)^2+12.718$

$2.503 \le x \le 3.537$, $y= -16(x-3)^2 + 12.616$

$3.537\le x \le 4.506$, $y= -16(x-4)^2 +11.428$

$4.506\le x \le 5.535$, $y= -16(x-5)^2 +11.24$

$5.535 \le x \le 6.508$, $y= -16(x-6)^2+10.132$

$6.508 \le x \le 7.532$, $y= -16(x-7)^2 +9.87$

$7.532 \le x \le 8.51$, $y= -16(x-8)^2 +8.83$

$8.51 \le x \le 9.531$, $y= -16(x-9)^2 +8.507$

$9.531 \le x \le 10.512$, $y= -16(x-10)^2 +7.523$

$10.512 \le x \le 11.529$, $y= -16(x-11)^2 +7.148$

$11.529 \le x \le 12.513$, $y= -16(x-12)^2+ 6.213$

$12.513 \le x \le 13.528$, $y= -16(x-13)^2 +5.793$

$13.528 \le x \le 14.503$, $y= -16(x-14)^2 +4.899$

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$14.503 \le x \le 15.548$, $y= -16(x-15)^2 +4.805$

Together, these model the vertical position over time graph (*Fig. 2)*. Then, in order to create the horizontal position over time graph, I found what point the final parabola touched the ground, and used that to calculate the rate of the horizontal position because it is a constantly increasing function. I determined this function to be y=0.75036446x (*Fig. 3)*. Then, I combined the two graphs to create the horizontal vs vertical position graph (*Fig. 4)*.

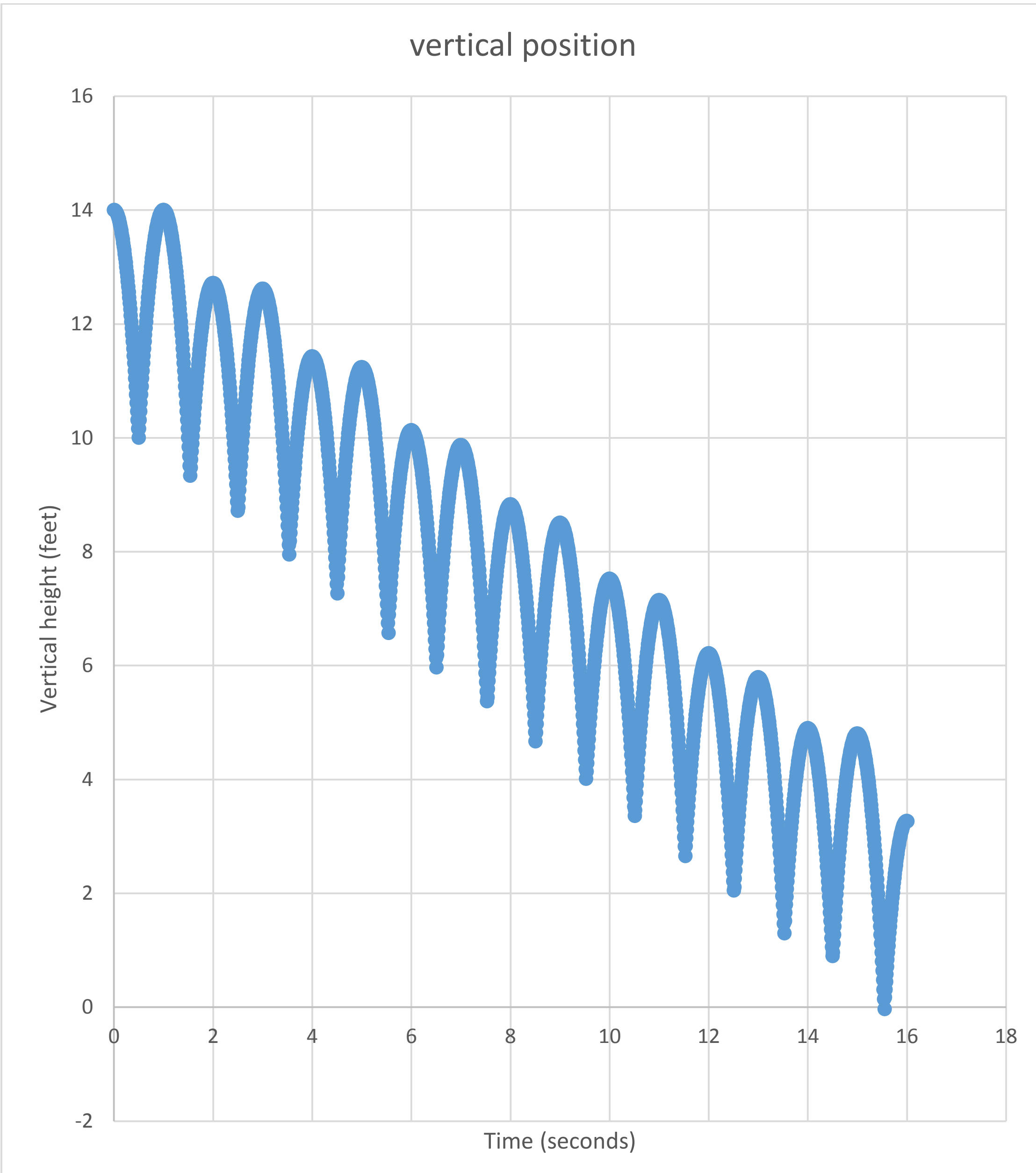

Figure 2

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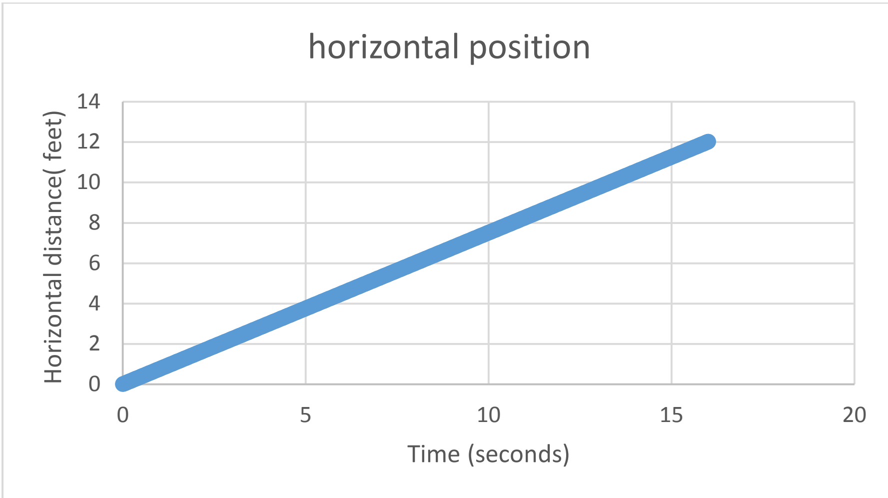

Figure 3

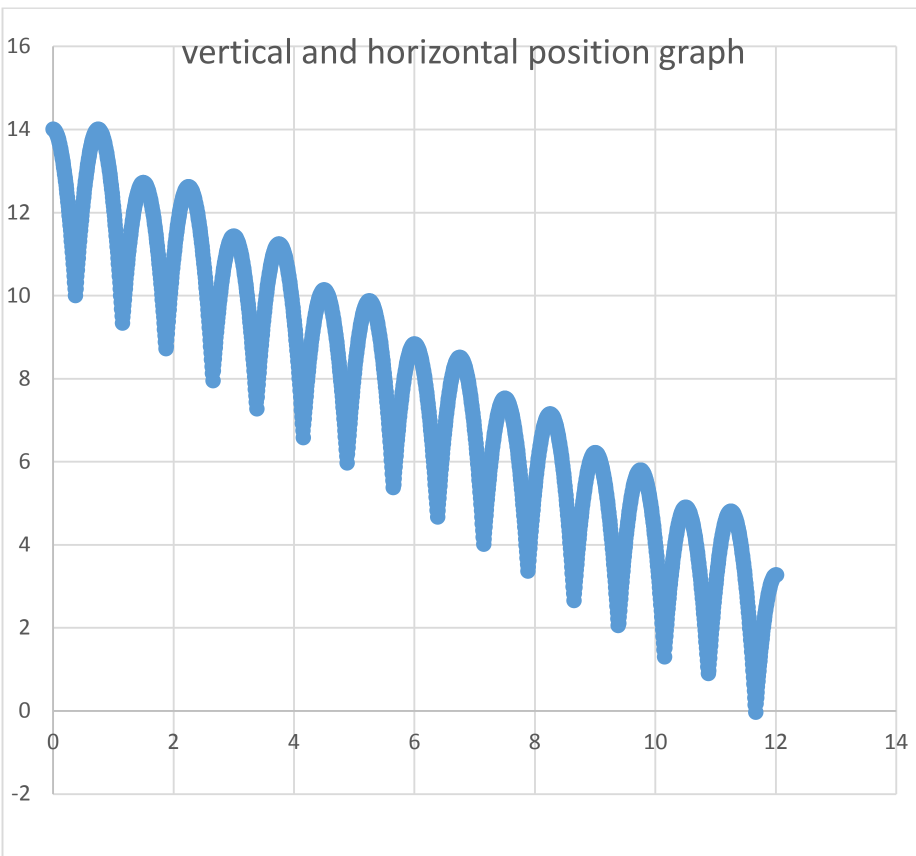

Figure 4

Then, to model the velocity of the graph, I took the derivative of each bounce parabola (*Fig. 5)*.

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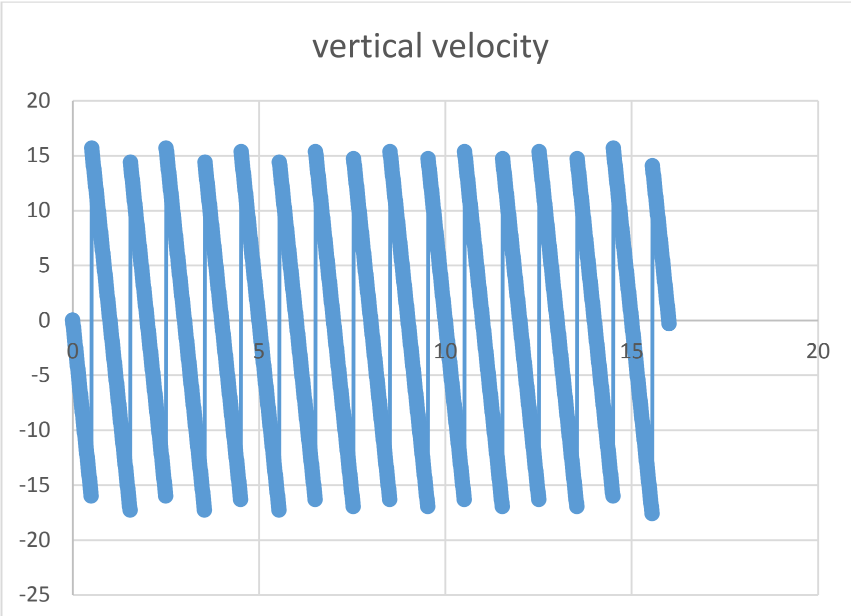

Figure 5

Then, I calculated the acceleration of each parabola, which equaled -32 feet per second per second for every parabola as anticipated (*Fig. 6)*.

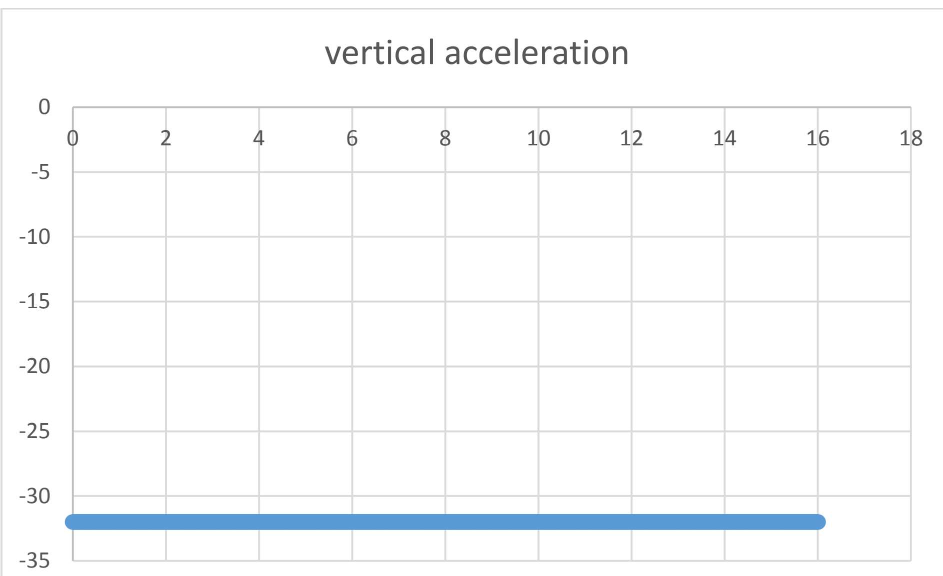

Figure 6

The vertical position graph shows the ball being let go from a height of four feet above the first stair step, and bouncing off that step on to the next ones. It shows that at 15.54 seconds, the ball reaches the ground. The horizontal position graph shows that the horizontal position is constantly increasing over time. Together, they demonstrate that the ball bounces off almost the middle of each step and eventually bounces off the stairs before the horizontal position reaches 12 feet because the run of the stairs is only 11 feet 8 inches. The vertical velocity graph shows that there is a sharp increase in velocity for each bounce followed by a constant decrease. This is the typical format for the velocity of a bouncing ball because of the separate parabolas it creates. The acceleration is constant because the only force acting upon the ball is acceleration due to earth's gravity.

There are several limitations to my findings. First, although this presents an ideal scenario of a ball being bounced down stairs, the act of bouncing a ball downstairs is far more complex. There are several variables that affect this that cannot be analyzed simply by using calculus because this is a very physics based problem. Because of the complexity of this problem, it is impossible to create a set of conditions that would guarantee an ideal bounce using only Calculus I principles. The information presented here can only model what would be happening if this ideal bouncing scenario were to actually happen. It is impossible to determine a singular bouncing angle, height, or force that would make this scenario occur.

The scenario of a ball bouncing down every step in a staircase is an interesting problem to examine, but has infinitely many solutions due to the complexity of a ball bouncing pattern. The problem can be graphically examined to better understand the mechanics of a ball bounce, however a singular solution is impossible to determine. This problem scenario is a good exercise in interpreting graphs and throughout it I became much more comfortable with the mechanics of a quadratic parabola, and the relation between position, velocity, and acceleration. These graphs also demonstrate the relationship between vertical and horizontal position and how time affects both, and how they cannot initially be graphed together.