Math 690 Spring 2004

Problem: Show that the operation of addition is a continuous function from $\mathbf{R} \times \mathbf{R}$ to $\mathbf{R}$.
Solution: The function is defined by $f(x,y) = x + y$. We prove that it is continuous at $(a,b) \in \mathbf{R} \times \mathbf{R}$.
Let $c = f(a,b) = a + b$. Given $\epsilon > 0$. Let $\delta = \epsilon/2$.
Suppose (x,y) lies within δ from (a,b). Then $\sqrt{(x-a)^2 + (y-b)^2} < \delta$. Hence $(x-a)^2 + (y-b)^2 < \delta^2$. Therefore $(x-a)^2 < \delta^2$ and $(y-b)^2 < \delta^2$. It follows that $|x-a| < \delta$ and $|y-b| < \delta$. Consider now $z = f(x,y) = x + y$. From the above inequalities and the triangle inequality it follows that

$$|z - c| = |(x+y) - (a+b)| = |(x-a) + (y-b)| \leq |x-a| + |y-b| < \delta + \delta = 2\delta = \epsilon.$$

We have proven that if $(x,y) \in D((a,b),\delta)$ then $z = f(x,y) \in D(c,\epsilon)$. Since $f(a,b) = c$, this means that $f(D((a,b),\delta)) \subset D(f(a,b),\epsilon)$. Therefore f is continuous at (a,b) and, since (a,b) was chosen arbitrarily, it is continuous on $\mathbf{R} \times \mathbf{R}$.