Contents

1. Lefschetz-Type Coincidence Theorems.

1.1. Introduction. A Lefschetz-type coincidence theorem states the following. Given a pair of continuous maps $f, g : X \to Y$, the Lefschetz number λ_{fg} of the pair (f, g) is equal to its coincidence index I_{fg}, while I_{fg} is defined in such a way that

$$I_{fg} \neq 0 \Rightarrow f(x) = g(x) \text{ for some } x \in X.$$

Thus, if the Lefschetz number, a computable homotopy invariant of the pair, is not zero, then there is a coincidence. We now consider two ways to define the coincidence index in two different settings.

Case 1.: Let M_1, M_2 be closed n-manifolds, X an open subset of M_1, N an open subset of M_1, V an open subset of M_2, $f, g : X \to V \subset M_2$ maps, and

$$\{x \in X : f(x) = g(x)\} \subset N \subset \overline{N} \subset X \subset M_1.$$

Then the *coincidence index* I^X_{fg} [**69**, p. 177] is the image of the fundamental class O_{M_1} of M_1 under the composition:

$$H_n(M_1) \xrightarrow{inclusion} H_n(M_1, M_1 \backslash N) \xrightarrow{excision} H_n(X, X \backslash N)$$
$$\xrightarrow{(f,g)_*} H_n(M_2 \times M_2, M_2 \times M_2 \backslash \delta(M_2)) \simeq \mathbf{Q},$$

where $\delta(x) = (x, x)$.

This definition represents the original approach to the coincidence problem for closed manifolds due to Lefschetz [**47**]. It was later generalized to the case of manifolds with boundary (and a boundary-preserving f) by Nakaoka [**53**], Davidyan [**12, 13**], Mukherjea [**52**].

Case 2.: Let V be an open subset of n-dimensional Euclidean space, $f : X \to V$ a Vietoris map (in particular, $f^{-1}(y)$ is acyclic for each $y \in V$), $g : X \to K$ a map, $K \subset V$ a finite polyhedron, and $N = f^{-1}(K)$.

Then the *coincidence index* $I(f, g)$ [**24**, p. 38] is the image of the fundamental class O_K of K under the composition:

$$H_n(V, V \backslash K) \xrightarrow{f_*^{-1}} H_n(X, X \backslash N) \xrightarrow{(f-g)_*} H_n(\mathbf{R}^n, \mathbf{R}^n \backslash \{0\}) \simeq \mathbf{Q}.$$

This coincidence index has evolved from the Hopf's fixed point index I_g ($X = V, f = Id_V$), see Brown [**8**, Chapter IV], Dold [**16**], [**19**, VII.5]. This approach was developed by Eilenberg and Montgomery [**21**], Begle [**3**], Gorniewicz and Granas [**24, 26, 27**] and others, see [**24**] for bibliography. It does not require any knowledge of the topology of X and, for this reason, is especially well suited for the study of fixed points of multivalued maps. For an acyclic-valued multifunction $F : Y \to Y$, we let X be the graph of F and f, g be the projections of X on Y, then f is a Vietoris map, and a coincidence of (f, g) is a fixed point of F. This construction can not be applied to Case 1 because the graph of a multifunction $F : M_2 \to M_2$ is not, in general, a manifold.

The restrictions on spaces and maps in Case 1 and Case 2 are necessary to ensure the existence of an appropriate homomorphism $f_! : H(Y) \to H(X)$, which we shall call here the transfer of f. Then the Lefschetz number of $\varphi_{fg} = g_* f_!$ is said to be the Lefschetz number of the pair (f, g). For Case 1, we let

$$f_! = D_1 f^* D_2^{-1},$$

where D_1 and D_2 are the Poincare duality isomorphisms for manifolds M_1 and M_2. For Case 2, we let

$$f_! = f_*^{-1},$$

with the existence of f_*^{-1} guaranteed by the Vietoris-Begle Mapping Theorem 4.4.

Until now these two ways to treat the same problem have been studied separately. In this chapter we provide a unified approach. We define the coincidence index as in Case 1, for arbitrary maps to an n-manifold (with or without boundary), $n \geq 1$, but with no restriction on their domain, as in Case 2. Roughly, we combine

$$\begin{array}{lll} \text{Case 1:} & n\text{-manifold} \xrightarrow{\text{any map}} & n\text{-manifold, and} \\ \text{Case 2:} & \text{any space} \xrightarrow{\text{Vietoris map}} & \text{Euclidean space,} \end{array}$$

into

$$\text{any space} \xrightarrow{\text{any map}} \text{manifold.}$$

Under the restrictions of Cases 1 and 2, our main theorem reduces to the results mentioned above (see Sections 4 and 10), but it also applies to the case

$$\text{non-manifold} \quad \xrightarrow{\text{any map}} \quad \text{manifold},$$

as well as

$$m\text{-manifold} \quad \xrightarrow{\text{any map}} \quad n\text{-manifold}, \ m \neq n$$

(see Section 5). For the sake of simplicity, we limit our attention to the case when Y is subset of a manifold, although some of the results can be extended to include spaces as general as ANR's.

An important particular situation when the choice of the transfer is obvious (see Corollary 5.1 and Theorem 9.1) occurs if the condition below is satisfied:

$$(\mathbf{A}) \ f_* : H_n(X, X\backslash N) \to H_n(V, V\backslash K) \text{ is nonzero}$$

(note that, in the case of two n-manifolds, the condition simply means that f has nonzero degree). This condition can be fairly easily verified for specific spaces and maps. In particular, we can relax the Vietoris condition on f or assume that f is a fibration (see Section 5). Furthermore, when $g_* = 0$ (in reduced homology) the Lefschetz number of $\varphi_{fg} = g_* f_!$ is equal to 1, so condition (A) implies the existence of a coincidence. For example, $f : (\mathbf{D}^2, \mathbf{S}^1) \to (\mathbf{S}^2, \{*\})$ has a coincidence with any $g : \mathbf{D}^2 \to \mathbf{S}^2$.

Our approach seems to be related to a suggestion made by Dold in [**18**]. The subject of his paper is coincidences on ENR_B's, Euclidean neighborhood retracts over space B, and will remain outside the scope of the present paper. In the end of his paper Dold compares his Theorem 2.1 to a result that assumes that one of the maps is Vietoris (Case 2): "It appears less general than 2.1 because 2.1 makes no acyclicity assumption ... On the other hand, it has a more general aspect than 2.1 because it does not assume an actual fibration (or ENR_B), only a "cohomology fibration" (with "pointlike" fibres). This comparison suggests a common generalization, namely to general *cohomology fibrations* ..." (cf. Section 9).

The proofs of our main theorems are self-contained and use some constructions from Gorniewicz [**24**, V.5.1, pp. 38-40] (see also Dold [**19**, VII.6, pp. 207-211]) and Vick [**69**, Chapter 6].

The chapter is organized as follows. In Section 2 we present our main results (Theorems 2.1 - 2.3) and in Section 3 we prove Theorem 2.1. In Section 4 we prove Theorem 2.3 (a Lefschetz-type coincidence theorem for maps to a manifold with boundary) and obtain Nakaoka's Coincidence Theorem for boundary-preserving maps between manifolds and Gorniewicz's Coincidence Theorem for Vietoris maps. Section 5 is devoted to applications of the main theorem and examples with the emphasis on situations that are not covered by the two traditions discussed above. Sections 6 - 8 contain the proof of Theorem 2.2 in a slightly more general setting (for maps to an open subset of a manifold). In the last two sections we prove a Lefschetz-type coincidence theorem for generalized Case 2.

1.2. Main Results. Let $E = \{E_q\}$ be a graded $\mathbf{Q}$-module with

$$\dim E_q < \infty, \ q = 0, \ldots, n, \quad E_q = 0, \ q = n+1, \ldots$$

(in other words, E is finitely generated). If $h = \{h_q\}$ is an endomorphism of E of degree 0, then the *Lefschetz number* $L(h)$ of h is defined by

$$L(h) = \sum_q (-1)^q tr(h_q),$$

where $tr(h_q)$ is the trace of h_q.

By H we denote the singular homology and by $\check{H}$ the Čech homology with compact carriers with coefficients in $\mathbf{Q}$. Throughout this chapter M is an oriented connected compact closed n-manifold, $n \geq 0$ (although most results remain valid for a non-orientable M if we take the coefficient field to be $\mathbf{Z}_2$).

Let X be a topological space, $N \subset X$, M be an oriented connected compact closed n-manifold, $(S, \partial S)$ a connected n-submanifold with (possibly empty) boundary ∂S and interior $\overset{\circ}{S} = S \backslash \partial S$. Let

$$f : (X, X \backslash N) \longrightarrow (S, \partial S), \quad g : X \longrightarrow S,$$

be continuous maps with $Coin(f, g) = \{x \in X : f(x) = g(x)\} \subset N$. Let

$$M^{\times} = (M \times M, M \times M \backslash \delta(M)),$$

where $\delta(x) = (x, x)$ is the diagonal map. Then the map $f \times g : (X, X \backslash N) \times X \to M^{\times}$ is well defined.

Fix an element $\mu \in H_n(X, X \backslash N)$.

The *coincidence index* I_{fg} *of the pair* (f, g) (with respect to μ) is defined by

$$I_{fg} = (f \times g)_* \delta_*(\mu) \in H_n(M^{\times}) \simeq \mathbf{Q}.$$

Let $O_S \in H_n(S, \partial S)$ be the fundamental class of $(S, \partial S)$. The *transfer of* f (with respect to μ) is the homomorphism $f_! : H(S) \to H(X)$ given by

$$f_! = (f^* D^{-1}) \frown \mu,$$

where $D : H^*(S, \partial S) \to H(S)$ is the Poincare-Lefschetz duality isomorphism. Then we define $\lambda_{fg} = L(g_* f_!)$ to be *the Lefschetz number of the pair* (f, g) (with respect to μ).

The proofs of the three theorems below are located in Sections 3, 8 and 4 respectively.

Theorem 1.1. *For a pair* $f : (X, X \backslash N) \to (S, \partial S), \quad g : X \to \overset{\circ}{S}$, *we have*

$$I_{fg} = I_*(Id \otimes g_* f_!) \delta_*(O_S),$$

where $I : (S, \partial S) \times \overset{\circ}{S} \to M^{\times}$ *is the inclusion.*

Theorem 1.2. *For any homomorphism* $\varphi : H(S) \to H(\overset{\circ}{S})$ *we have*

$$L(\varphi i_*) = I_*(Id \otimes \varphi) \delta_*(O_S),$$

where $i : \overset{\circ}{S} \to S$ *is the inclusion.*

The following is the main theorem of the chapter.

Theorem 1.3 (Lefschetz-Type Coincidence Theorem). *For any pair* $f : (X, X \backslash N) \to (S, \partial S)$, $g : X \to S$, *the coincidence index is equal to the Lefschetz number (with respect to* μ*):*

$$I_{fg} = L(g_* f_!).$$

Moreover, if $L(g_* f_!) \neq 0$, *then* (f, g) *has a coincidence.*

If $(X, X\backslash N)$ is a manifold with boundary, we get the Lefschetz-type coincidence theorem for Case 1 by letting μ be its fundamental class. To get such a theorem for Case 2 we let $M = \mathbf{R}^n \cup \{\infty\}$ and $\mu = f_*^{-1}(O_S)$.

Now we consider these results in detail.

1.3. The Transfer of a Map. Recall [**69**, p. 156] that if $(S, \partial S)$ is a compact oriented n-manifold, then the Poincare-Lefschetz duality isomorphism

$$D : H^{n-k}(S, \partial S) \longrightarrow H_k(S)$$

is given by $D(a) = a \frown O_S$. Suppose $f : (S_1, \partial S_1) \to (S_2, \partial S_2)$, where $(S_i, \partial S_i)$, $i = 1, 2$, are n-manifolds, is a map. Following Vick [**69**, Chapter 6] we could define $f_!$ as follows. If

$$D_i : H^{n-k}(S_i, \partial S_i) \longrightarrow H_k(S_i), \quad i = 1, 2,$$

denote the duality isomorphisms, we let

$$f_! = D_1 f^* D_2^{-1},$$

so that $f_!$ is the composition of the following maps:

$$H_k(S_2) \xrightarrow{D_2^{-1}} H^{n-k}(S_2, \partial S_2) \xrightarrow{f^*} H^{n-k}(S_1, \partial S_1) \xrightarrow{D_1} H_k(S_1).$$

Similarly we define $f_!$ for $f : (X, X\backslash N) \to (S, \partial S)$, where X is an arbitrary topological space: the transfer of f is the homomorphism $f_! : H(S) \to H(X)$ given by

$$f_! = (f^* D_2^{-1}) \frown \mu,$$

where $D_2 : H^*(S, \partial S) \to H(S)$ is the Poincare-Lefschetz duality isomorphism.

To prove Theorem 2.1 we will use some arguments from Vick [**69**, pp. 184-186]. Select a basis $\{x_i\}$ for $H^*(S)$ and denote by $\{a_i\}$ the basis for $H(S)$ dual to $\{x_i\}$ under the Kronecker index. Define a basis $\{x_i'\}$ for $H^*(S, \partial S)$ by requiring that $D_2(x_i') = a_i$ and let $\{a_i'\}$ be the basis for $H(S, \partial S)$ dual to $\{x_i'\}$ under the Kronecker index. Thus we have

$$< x_i, a_j >=< x_i', a_j' >= \delta_{ij},$$
$$D_2(x_i') = x_i' \frown O_S = a_i.$$

Similarly, select a basis $\{y_i'\}$ for $H^*(X, X\backslash N)$ and denote by $\{b_i'\}$ the basis for $H(X, X\backslash N)$ dual to $\{y_i'\}$ under the Kronecker index. We define the homomorphism $D_1 : H^*(X, X\backslash N) \to H(X)$ by $D_1(x) = x \frown \mu$ and we let $b_i = D_1(y_i')$. Next we let $\{y_i\} \subset H^*(X)$ be a collection dual to $\{b_i\}$ under the Kronecker index. Thus we have

$$< y_i, b_j >=< y_i', b_j' >= \delta_{ij},$$
$$D_1(\ y_k') = y_i' \frown \mu = b_i.$$

LEMMA 1.4. *(cf. Vick* [**69**, Lemma 6.10, p. 185]*)*

$$\sum_i (Id \times f_!)(a_i' \times a_i) = \sum_i (f_* \times Id)(b_i' \times b_i).$$

Proof. Since $\{y_i'\}$ and $\{a_i'\}$ are bases, there are representations

$$f^*(x_i') = \textstyle\sum_k \gamma_{ik} y_k' \text{ and } f_*(b_j') = \sum_k \lambda_{kj} a_k'.$$

Then we have

$$\gamma_{ij} =< \sum_k \gamma_{ik} y_k', b_j' >=< f^*(x_i'), b_j' >=< x_i', f_*(b_j') >=< x_i', \sum_k \lambda_{kj} a_k' >= \lambda_{ij},$$

so the following holds:

$$f_*(b'_i) = \sum_k \gamma_{ki} a'_k.$$

Next, consider

$$D_1 f^* D_2^{-1}(a_i) = D_1 f^*(x'_i) = D_1 \sum_k \gamma_{ik} y'_k = \sum_k \gamma_{ik} b_k.$$

Therefore, we have

$$\begin{aligned}(Id \times f_!)(a'_i \times a_i) \quad &= a'_i \times D_1 f^* D_2^{-1}(a_i)\\ &= a'_i \times \textstyle\sum_k \gamma_{ik} b_k\\ &= \textstyle\sum_k \gamma_{ik}(a'_i \times b_k).\end{aligned}$$

On the other hand, consider

$$\begin{aligned}(f_* \times Id)(b'_i \times b_i) \quad &= f_*(b'_i) \times b_i\\ &= (\textstyle\sum_k \gamma_{ki} a'_k) \times b_i\\ &= \textstyle\sum_k \gamma_{ki}(a'_k \times b_i).\end{aligned}$$

Therefore summation over i produces the same result. □

LEMMA 1.5. *(cf. Vick* [**69**, Lemma 6.11, p. 186]*)*

(a) $\delta_*(O_S) = \sum_i (a'_i \times a_i)$,
(b) $\delta_*(\mu) = \sum_i (b'_i \times b_i)$.

Proof. (a) By the Künneth formula, $\{a'_i \times a_j\}$ is a basis of $H((S, \partial S) \times S)$, and $\{x'_i \times x_j\}$ is the dual basis of $H^*((S, \partial S) \times S)$. Then the identity follows from the equations:

$$\begin{aligned}< x'_j \times x_k, \delta_*(O_S) > \quad &=< \delta^*(x'_j \times x_k), O_S >\\ &=< x'_j \smile x_k, O_S >\\ &=< x_k, x'_j \frown O_S >\\ &=< x_k, a_j >\\ &= \delta_{kj.}\end{aligned}$$

(b) It is clear that $\delta_*(\mu)$ belongs to the subspace of $H((X, X\backslash N) \times X)$ spanned by $\{b'_i \times b_i\}$. Then the rest of the proof follows (a). □

Proof of Theorem 2.1. We have

$$\begin{aligned}I_{fg} = (f \times g)_* \delta_*(\mu) \quad &= I_*(Id \times g_*)(f_* \times Id) \textstyle\sum_i (b'_i \times b_i) && \text{by Lemma 3.2}\\ &= I_*(Id \times g_*)(Id \times f_!) \textstyle\sum_i (a'_i \times a_i) && \text{by Lemma 3.1}\\ &= I_*(Id \times g_* f_!) \delta_*(O_S) && \text{by Lemma 3.2.}\square\end{aligned}$$

Thus I_{fg} is the image of O_S under the composition of the following maps:

$$H(S, \partial S) \xrightarrow{\delta_*} H(S, \partial S) \otimes H(S) \xrightarrow{Id \otimes \varphi} H(S, \partial S) \otimes H(\overset{\circ}{S}) \xrightarrow{I_*} H(M^\times),$$

while $\varphi = g_* f_!$ is defined by the following diagram:

$$\begin{array}{ccc} H^*(X, X\backslash N) & \xleftarrow{\quad f^* \quad} & H^*(S, \partial S)\\ \downarrow^{\frown \mu} & & \downarrow^{D_2}\\ H(X) & \xrightarrow{\quad g_* \quad} & H(S). \end{array}$$

It is worth mentioning that Nakaoka [**53**] defines the *coincidence transfer homomorphism* τ_{fg} for a pair of fiber-preserving maps $f, g : E \to E'$, where E, E' are manifolds with boundary. If the base is trivial then, according to [**53**, Theorem 5.1 (ii)], τ_{fg} is related to φ as follows:

$$\tau_{fg}(1) = L(\varphi_{fg}).$$

1.4. The Main Theorem and Cases 1 and 2. The sum of Theorems 2.1 and 2.2 gives us a Lefschetz-type coincidence theorem for $g : X \to \overset{\circ}{S}$. To prove the statement for $g : X \to S$ (Theorem 2.3) we need the following fact.

PROPOSITION 1.6 (Topological Collaring Theorem). [**69**, Theorem 5.2, p. 154] *Let $(S, \partial S)$ be a manifold. Then there is a manifold $(T, \partial T)$ obtained from $(S, \partial S)$ by attaching a "collar":*

$$T = S \cup (\partial S \times [0, 1]).$$

Next we restate and prove Theorem 2.3.

THEOREM 1.7. *For any pair $f : (X, X \backslash N) \to (S, \partial S)$, $g : X \to S$, with $Coin(f, g) \subset N$, the coincidence index is equal to the Lefschetz number (with respect to μ):*

$$I_{fg} = L(g_* f_!).$$

Moreover, if $L(g_* f_!) \neq 0$, then (f, g) has a coincidence.

Proof. We attach "collars" to X and S as follows. Let

$$Z = X \cup ((X \backslash N) \times [0, 1]),$$

such that $X \cap ((X \backslash N) \times [0, 1]) = (X \backslash N) \times \{0\}$. And we assume that according to the proposition above

$$T = S \cup (\partial S \times [0, 1]) \subset M$$

is an n-manifold. Then we define $G : Z \to \overset{\circ}{T}$ by (cf. [**13**])

$$G = jgr,$$

where $j : S \to \overset{\circ}{T}$ is the inclusion, $r : Z \to X$ is the retraction. We also define $F : (Z, (X \backslash N) \times \{1\}) \to (T, \partial T)$ by

$$\begin{array}{ll} F(x, t) = (f(x), t) & \text{if } (x, t) \in (X \backslash N) \times [0, 1], \\ F(x) = f(x) & \text{if } x \in X. \end{array}$$

From Theorems 2.1 and 2.2 we have

$$I_{FG} = L(G_* F_! i_*),$$

where $i : \overset{\circ}{T} \to T$ is the inclusion. The inclusions and retractions induce isomorphisms, $F_* = f_*$, $G_* = g_*$, so $L(G_* F_! i_*) = L(g_* f_!)$. Next, the following diagram commutes:

$$\begin{array}{ccc} (X, X \backslash N) & \xrightarrow{(f,g)} & M^{\times} \\ \uparrow^{r} & & \| \\ (Z, (X \backslash N) \times \{1\}) & \xrightarrow{(F,G)} & M^{\times}, \end{array}$$

which means that $I_{FG} = I_{fg}$. Thus, we have

$$I_{fg} = I_{FG} = L(G_* F_! i_*) = L(g_* f_!). \ \square$$

Halpern [**32**] proves a Lefschetz-type coincidence theorem in an even more general situation: he considers $f, g : X \to Y$, where both X and Y are arbitrary topological spaces. His Lefschetz number is the Lefschetz number of the homomorphism $\varphi : Y \to Y$ given by

$$\varphi(z) = g_*(f^*(b/z) \frown a)$$

for some elements $a \in H_n(X)$ and $b \in H^n(Y \times Y)$, and he proves that $L(\varphi) \neq 0$ implies that $Coin(f, g) \neq \emptyset$. To compare his result with ours, observe first that he does not define the coincidence index, which has an independent interest, and second his theorem does not include the Brouwer fixed point theorem.

1.4.1. *Case 1.* To get a Lefschetz-type coincidence theorem for Case 1, i.e., when $(X, X\backslash N)$ is an n-manifold, we simply let μ be fundamental class of that manifold.

Corollary 1.8. *Let $(S_1, \partial S_1), (S_2, \partial S_2)$ be oriented compact connected n-manifolds, and let*

$$f : (S_1, \partial S_1) \longrightarrow (S_2, \partial S_2), \quad g : S_1 \longrightarrow S_2$$

be continuous maps. If $Coin(f, g) \cap \partial S_1 = \emptyset$, then the coincidence index with respect to $\mu = O_{S_1}$ is equal to the Lefschetz number:

$$I_{fg} = L(g_* f_!)$$

(here $f_!$ is defined via Poincare duality for S_1 and S_2). Moreover, if $L(g_* f_!) \neq 0$ then (f, g) has a coincidence.

Several authors have dealt with a Lefschetz-type coincidence theorem for manifolds with boundary. Corollary 4.3 provides little additional information in comparison to these results but still has certain advantages.

The Lefschetz-Nakaoka Coincidence Theorem [**9**, Theorem 3.2] (it is Lemma 8.1 combined with Theorem 5.1 of Nakaoka [**53**]) is identical to our Theorem 4.3 but applies only to manifolds with nonempty boundary. The reason for this is that in [**53**] manifolds with boundary are "doubled" (two copies are glued together along the boundary) and then Nakaoka's Lefschetz-type coincidence theorem for closed manifolds is applied. Of course, the case of empty boundary follows from the classical Lefschetz coincidence theorem [**69**, Theorem 6.13], but the case $\partial S_1 = \emptyset$, $\partial S_2 \neq \emptyset$ is still excluded. Bredon [**6**, VI.14] also considers manifolds with empty and nonempty boundary separately. Theorem of Davidyan [**13**] and Theorem 2.1 of Mukherjea [**52**] use collaring instead of doubling, so they can be specialized to manifolds with empty boundary. But they do not prove that the coincidence index is equal to the Lefschetz number (Davidyan [**12**] proves this identity only for manifolds without boundary).

Therefore Corollary 4.3 is of some interest, because it opens a possibility of defining a coincidence index for all manifolds with boundary, empty or not. Such an index may be used for a unified Nielsen coincidence theory, see Brown and Schirmer [**9, 10**], where boundaries are required to be nonempty.

1.4.2. *Case 2.* We can also obtain a Lefschetz-type coincidence theorem for Case 2 (cf. Theorem 10.5) with an additional assumption.

Recall [**24**, p. 13] that a map $f : (X, X_0) \to (Y, Y_0)$ is said to be *Vietoris* if (i) f is proper, i.e., $f^{-1}(B)$ is compact for any compact $B \subset Y$, (ii) $f^{-1}(Y_0) = X_0$, (iii) the set $f^{-1}(y)$ is acyclic with respect to the Čech homology for every $y \in Y$.

Proposition 1.9 (Vietoris-Begle Theorem). [**24**, p. 14] *If $f : (X, X_0) \to (Y, Y_0)$ is a Vietoris map, then $f_* : \check{H}(X, X_0) \to \check{H}(Y, Y_0)$ is an isomorphism.*

Corollary 1.10. *Suppose X is a topological space, $S \subset \mathbf{R}^n$ is a compact n-manifold. Suppose*

$$f, g : X \to \overset{\circ}{S}$$

are two continuous maps such that f is Vietoris. If $L(g_* f_*^{-1}) \neq 0$ with respect to the Čech homology over $\mathbf{Q}$, then the pair (f, g) has a coincidence.

Proof. We put $\mu = f_*^{-1}(O_S)$ and apply Theorem 4.2. Then we have $I_{fg} = L(g_* f_!)$. But by Theorem 4.4, f_*^{-1} exists, hence by the proposition below, we have $f_! = f_*^{-1}$. Therefore $I_{fg} = L(g_* f_*^{-1})$. □

Proposition 1.11. *(cf.* [**6**, Proposition VI.14.1 (6), p. 394]*) If $f_*(\mu) = O_S$ then $f_* f_! = Id$.*

1.5. Corollaries and Examples. Suppose (X, X') is a topological space, $(S, \partial S)$ is an oriented compact connected n-manifold. Consider the following condition:

(A): $f_* : H_n(X, X') \to H_n(S, \partial S)$ is a nonzero homomorphism.

First we will consider analogues of some well-known theorems about maps between manifolds without the assumption that the domain of the maps is a manifold. The following is a generalization of Theorem 2.2 of Mukherjea [**52**].

Corollary 1.12. *Suppose $f : (X, X') \to (S, \partial S)$ satisfies condition (A) and $g : X \to S$ induces $g_* = 0$ (in reduced homology). Then (f, g) has a coincidence.*

Proof. First we select $\mu \in H_n(X, X')$ such that $f_*(\mu) = O_S$. Then by Proposition 4.6, we have $f_* f_! = Id$. Then $g_* f_! : H_i(S, \partial S) \to H_i(S, \partial S)$ is non-zero for $i = 0$ and zero for $i \neq 0$. Therefore $L(g_* f_!) = 1$, so there is a coincidence by Theorem 2.3. □

Corollary 5.1 allows us to use a version of the Vietoris Theorem stronger than Proposition 4.4 (see also Section 5): even if $H_n(f^{-1}(x)) \neq 0$ for some x, while $H_i(f^{-1}(x)) = 0$ for all x and $0 \leq i < n$, we still have an epimorphism $f_* : H_n(X, X \backslash N) \to H_n(S, \partial S)$. Therefore condition (A) is satisfied. For other versions of the Vietoris Theorem see [**44**].

The following is a generalization of the Kronecker theorem: a map with nonzero degree is onto.

Corollary 1.13. *If $f : (X, X') \to (S, \partial S)$ satisfies condition (A) then f is onto.*

Proof. For a given $y \in S$, we define $g : X \to S$ by $g(x) = y$, for all $x \in X$. Therefore $g_* : H(X) \to H(S)$ is a zero homomorphism, hence there is a coincidence by the previous corollary. Therefore y belongs to $f(X)$, so $f(X) = S$. □

A map $f : (S_1, \partial S_1) \to (S_2, \partial S_2)$, where $(S_i, \partial S_i)$, $i = 1, 2$, are manifolds, is called *coincidence-producing* [**9**, Section 7] if every map $g : S_1 \to S_2$ has a coincidence with f. Brown and Schirmer [**9**, Theorem 7.1] showed that if S_2 is acyclic, $n \geq 2$, then f is coincidence-producing if and only if $f_* : H_n(S_1, \partial S_1) \to H_n(S_2, \partial S_2)$ is nonzero. We call a map $f : (X, X') \to (S, \partial S)$ *weakly coincidence-producing* if every map $g : X \to S$ with $g_* = 0$ has a coincidence with f. Then Corollary 5.1 takes the following form.

Corollary 1.14. *If $f : (X, X') \to (S, \partial S)$ satisfies condition (A) then f is weakly coincidence-producing.*

For an acyclic S, this is a generalization of the "if" part of the Brown-Schirmer statement.

We conclude with a few examples of applications of Corollary 5.3. These examples are not included in either Case 1 or Case 2.

1.5.1. *Manifolds.* It is hard to come by an example of coincidences that does not involve manifolds. Yet we can consider a pair (X, X') such that X is a manifold (possibly with boundary) but (X, X') is not a manifold with boundary, i.e., X' is not the boundary of X (nor homotopically equivalent to it). This provides a setting not included in Case 1.

EXAMPLE 1.15. Let $f : (\mathbf{D}^2, \partial e \cup \partial e') \to (\mathbf{D}^2, \mathbf{S}^1)$, where e and e' are disjoint cells in $\mathbf{D}^2$, be a map.

Then it is a matter of simple computation to check whether condition (A) is satisfied. For examples of acyclic manifolds, see Brown and Schirmer [**9**, Section 7].

EXAMPLE 1.16. Let $f : (\mathbf{I}, \partial \mathbf{I}) \times \mathbf{S}^1 \to (\mathbf{D}^2, \mathbf{S}^1 \cup \{0\}),\ \mathbf{I} = [0, 1]$, be the map that takes by identification $\{0\} \times \mathbf{S}^1$ to $\{0\}$.

It is clear that condition (A) is satisfied for $X' = \{1\} \times \mathbf{S}^1$, therefore by Corollary 5.3, any map homotopic to f has a coincidence with a map g if $g_* = 0$. But $f^{-1}(0) = \mathbf{S}^1$ is not acyclic, so this example is not covered by Case 2 and Gorniewicz's Theorem. On the other hand, even though f maps a manifold to a manifold, it does not map boundary to boundary. Therefore, Case 1 and the results discussed in Section 4 do not include this example.

In a similar fashion we can show that the projection of the torus $\mathbf{T}^2$ on the circle $\mathbf{S}^1$ is a weakly coincidence-producing map. This is an example of a map between manifolds of different dimensions. For a negative example of this kind, take the Hopf map $f : \mathbf{S}^3 \to \mathbf{S}^2$. Then for any g, the Lefschetz number and the coincidence index of the pair (f, g) are equal to zero.

1.5.2. *Non-manifolds.* Let E be a space that is not acyclic and not a manifold, e.g., the "figure eight". Consider the projection $f : (X, X') = (\mathbf{I}, \partial \mathbf{I}) \times E \to (\mathbf{I}, \partial \mathbf{I}), \mathbf{I} = [0, 1]$, onto the first coordinate. Then f clearly satisfies (A). Thus f is weakly coincidence-producing by Corollary 5.3. Observe that $f^{-1}(x) = E$ is not acyclic and X is not a manifold.

A relevant example is given in Kahn [**41**]. He constructed an infinite dimensional acyclic space X and an essential map $f : X \to \mathbf{S}^3$. Then f satisfies condition (A) and, since any $g : X \to \mathbf{S}^3$ induces a zero homomorphism in reduced homology, it follows that f is weakly coincidence-producing by Corollary 5.3.

1.5.3. *Fibrations and UV^n-maps.*

COROLLARY 1.17. *Suppose X is a topological space, M is an oriented compact closed $(n-1)$-connected n-manifold, $f : X \to M$ is a map, and*

: *(A′) $f_\# : \pi_n(X) \to \pi_n(M)$ is onto.*

Then f is weakly coincidence producing.

Proof. As M is $(n-1)$-connected, the Hurewicz homomorphism $h_n : \pi_n(M) \to H_n(M)$ is onto [**6**, p. 488]. Hence f_* is onto and (A) is satisfied. □

Condition (A′) holds when f is a fibration with $\pi_{n-1}(f^{-1}(y)) = 0$ (it follows from the homotopy sequence of the fibration [**6**, Theorem VII.6.7, p. 453]).

Condition (A′) also holds when f is onto and for each $y \in Y$, $f^{-1}(y)$ has the *UV^n-property* for each n (see [**44**, Section 4] and its bibliography): for any neighborhood U of $f^{-1}(x)$ there is a neighborhood $V \subset U$ such that any singular k-sphere in V is inessential in U, $0 \leq k \leq n$. Then the corollary gives a version of Theorem 1.2 of Gutev [**28**]. For related results see also [**20**, Section 2].

1.5.4. *m-Acyclic Maps.* A multivalued map $\Phi : Y \to Y$ is called *m-acyclic*, $m \geq 1$, if for each $x \in Y$, the set $\Phi(x)$ consists of exactly m acyclic components. Schirmer [**61**] proved that if Y is locally connected and simply connected then the graph X of Φ is a disjoint union of graphs X_i of m acyclic maps Φ_i, $i = 1, 2, ..., m$. Then the projections $f_i : X_i \to Y$ on the first coordinate induce isomorphisms, therefore condition (A) holds for f the projection of X onto Y.

Patnaik [**59**] defines an *m-map* Φ as a multifunction such that $\Phi(x)$ contains exactly m points for each x. Then he considers the Lefschetz number of a homomorphism similar to $g_* f_!$, although it is not clear how it is related to ours.

1.5.5. *Spherical Maps.* Continuing the discussion in the beginning of this section, what if the acyclicity condition for f fails at degree $(n-1)$? Then there is no version of the Vietoris Theorem available to ensure condition (A).

Definition 1.18. (cf. [**55, 25, 14**]) Let $B(A)$, where $A \subset \mathbf{R}^n$, denote the bounded component of $\mathbf{R}^n \backslash A$. Then a closed-valued u.s.c. map $\Phi : \mathbf{D}^n \to \mathbf{D}^n$ is called $(n-1)$*-spherical*, $n > 1$, if

: (i) for every $x \in \mathbf{D}^n$, $H(\Phi(x)) = H(\mathbf{S}^{n-1})$ or $H(point)$,
: (ii) for every $x \in \mathbf{D}^n$, if $x \in B(\Phi(x))$ then there exists an ε-neighborhood $O_\varepsilon(x)$ of x such that $x' \in B(\Phi(x'))$ for each $x' \in O_\varepsilon(x)$.

Corollary 1.19. *An $(n-1)$-spherical map $\Phi : \mathbf{D}^n \to \mathbf{D}^n$, $n > 1$, has a fixed point.*

Proof. We notice, first, that if Φ has no fixed points and there are no points x such that $x \in B(\Phi(x))$, then by replacing $\Phi(x)$ with $\Phi'(x) = \Phi(x) \cup B(\Phi(x))$ we obtain an acyclic multifunction without fixed points. Therefore we suppose that such an x exists and for simplicity assume that it is 0. Now, if 0 is not a fixed point, then from the upper semicontinuity of Φ and (ii) above, it follows that there is an $\varepsilon > 0$ such that

$$|x| < \varepsilon \Rightarrow x \in B(\Phi(x)) \text{ and } |\Phi(x)| > 2\varepsilon.$$

Let X be the graph of Φ, $K = \{x : |x| \geq 2\varepsilon\}$, and f, g the projections of X, and let $X' = f^{-1}(\mathbf{D}^n \backslash K)$. One can see that f is essentially the same as the projection of $(\mathbf{D}^n, \mathbf{S}^{n-1}) \times \mathbf{S}^1$ onto $(\mathbf{D}^n, \mathbf{S}^{n-1})$, and, therefore, induces a surjection

$$f_* : H_n((\mathbf{D}^n, \mathbf{S}^{n-1}) \times \mathbf{S}^1) \to H_n(\mathbf{D}^n, \mathbf{S}^{n-1}).$$

Hence condition (A) is satisfied, so by Corollary 5.3, Φ has a fixed point. □

Example 1.20. (O'Neill [**55**]) Let $\Phi : \mathbf{D}^2 \to \mathbf{D}^2$ be given by

$$\Phi(x) = \{y \in \mathbf{D}^2 : |y - x| = \rho(x)\} \cup \{y \in \mathbf{S}^1 : |y - x| > \rho(x)\},$$

where $\rho(x) = 1 - |x| + |x|^2$, $x \in \mathbf{D}^2$.

This example shows that condition (ii) of the above definition is necessary for existence of a fixed point.

1.6. The Coincidence Index of a Pair of Maps. The next three sections are devoted to the proof of Theorem 2.2, which will be carried out in a setting slightly more general than that of Section 2.

Consider the sets $K \subset V \subset M$. Assume $(M, V, M\backslash K)$ is an excisive triad, i.e., the inclusion $j : (V, V\backslash K) \to (M, M\backslash K)$ induces an isomorphism in homology. Let $i : K \to V,\ I : M \times K \to M^{\times}$ be the inclusions.

We start by recalling some facts about manifolds. The following definition and propositions are taken from Vick [**69**, Chapter 5].

Proposition 1.21. [**69**, Corollary 5.7, p. 136] *For each $p \in M$, the homomorphism*

$$i_{p*} : H_n(M) \longrightarrow H_n(M, M\backslash\{p\}) = T_p \simeq \mathbf{Q},$$

where $i_p : M \to (M, M\backslash\{p\})$ is the inclusion, is an isomorphism.

Definition 1.22. [**69**, p. 139] *The fundamental class of M* is an element $z \in H_n(M)$ such that

$$i_{p*} : H_n(M) \longrightarrow H_n(M, M\backslash\{p\}) = T_p$$

has $i_{p*}(z)$ a generator of T_p for each $p \in M$.

Proposition 1.23. [**69**, Lemma 5.12, p. 143] *We have an isomorphism*

$$\xi : \mathbf{Q} \simeq H_0(M) \simeq H_n(M^{\times}),$$

which sends the 0-chain represented by $p \in M$ into the relative class represented by $l_{p*}(s(p))$, where

$$l_{p*} : H_n(M, M\backslash\{p\}) \longrightarrow H_n(M^{\times})$$

is induced by $l_p(x) = (x, p),\ x \in M$, and $s : M \to T = \bigcup_{p \in M} T_p$ is the orientation map of M, with $s(p)$ a generator of T_p, for each $p \in M$.

Proposition 1.24. [**69**, Theorem 5.10, p. 140] *If s is the orientation map then there is a unique fundamental class $O_M \in H_n(M)$ such that $i_{p*}(O_M) = s(p)$ for each $p \in M$.*

Consider

$$M \xrightarrow{k} (M, M\backslash K) \xleftarrow{j} (V, V\backslash K),$$

where k is the inclusion.

Definition 1.25. (cf. [**24**, p. 16], [**19**, p. 192]) *The fundamental class O_K of the pair* $(V, V\backslash K)$ is defined by

$$O_K = j_{*n}^{-1} k_{*n}(O_M) \in H_n(V, V\backslash K).$$

Let X be a topological space, N a subset of X, and let

$$f, g : X \longrightarrow V,$$

be continuous maps. Suppose

$$Coin(f, g) \subset N.$$

Then the map $f \times g : (X, X\backslash N) \times X \to M^{\times}$ is well defined.

Fix an element $\mu \in H_n(X, X\backslash N)$.

Definition 1.26. The *coincidence index I_{fg} of the pair* (f, g) (with respect to μ) is defined by

$$I_{fg} = (f \times g)_* \delta_*(\mu) \in H_n(M^{\times}) \simeq \mathbf{Q}.$$

In the setting of Case 1 this definition turns into the usual one (cf. [**69**, p. 177]). Another observation: As I_{fg} is defined via a homomorphism from $H_n(X, X\backslash N)$ to $\mathbf{Q}$, it is an element of $H^n(X, X\backslash N)$.

This definition also includes the coincidence index for Case 2, as given in the proposition below.

PROPOSITION 1.27. *Suppose U is an open subset of n-dimensional Euclidean space, $f : X \to U$ a Vietoris map, $g : X \to K$ a map, where $K \subset U$ is a finite polyhedron. Let $N = f^{-1}(K)$ and $\mu = f_*^{-1}(O_K) \in H_n(X, X \backslash N)$. Then $I(f, g) = I_{fg}$.*

Proof. We identify $\mathbf{R}^n$ with a hemisphere of $M = \mathbf{S}^n$. Then from Proposition 6.3 it follows that $H_0(\mathbf{R}^n) \simeq H_n(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^n \backslash \delta(\mathbf{R}^n))$ under the homomorphism ξ sending the 0-chain represented by $p \in \mathbf{R}^n$ into the relative class represented by $l_{p*}(s(p))$. Therefore we have a commutative diagram

$$\begin{array}{ccccc}
H_0(\mathbf{R}^n) & \xrightarrow{\xi} & H_n((\mathbf{R}^n)^\times) & \xrightarrow{d_*} & H_n(\mathbf{R}^n, \mathbf{R}^n \backslash \{0\}) \\
\downarrow & & \downarrow & \nwarrow^{(f,g)_*} & \uparrow^{(f-g)_*} \\
H_0(M) & \xrightarrow{\xi} & H_n(M^\times) & \xleftarrow{(f,g)_*} & H_n(X, X \backslash N),
\end{array}$$

where the two vertical arrows are isomorphisms induced by the inclusions, $d(x, y) = x - y$. Since d_* is an isomorphism [**19**, Lemma VII.4.13, p. 200], it follows that

$$I_{fg} = (f, g)_*(\mu) = (f - g)_*(\mu) = (f - g)_* f_*^{-1}(O_K) = I(f, g).\square$$

To justify its name, a coincidence index should satisfy the property below.

LEMMA 1.28. *If $I_{fg} \neq 0$ (with respect to some μ) then the pair (f, g) has a coincidence.*

Proof. Suppose not, then $C = Coin(f, g) = \emptyset$. Hence $H_n((X, X \backslash C) \times X) = 0$. But the following diagram is commutative:

$$\begin{array}{ccc}
H_n((X, X \backslash N) \times X) & \xrightarrow{(f \times g)_*} & \mathbf{Q} \\
\downarrow^{k_*} & & \| \\
H_n((X, X \backslash C) \times X) & \xrightarrow{(f \times g)_*} & \mathbf{Q},
\end{array}$$

where k is the inclusion, so $I_{fg} = 0$. $\square$

1.7. Generalized Dold's Lemma. In this section we obtain a generalization of Dold's Lemma [**19**, Lemma VII.6.13, p. 210] (see also [**8**, p. 153]), which is necessary for our definition of the coincidence index. It is proved for singular homology, but when V is open and K is compact, we can interpret this result for Čech homology with compact carriers, as in [**24**, I.5.6, p. 17].

We define the following functions:

the transposition $t : V \times K \to K \times V$ by

$$t(x, y) = (y, x);$$

the scalar multiplication $m : \mathbf{Q} \otimes H(V) \to H(V)$ by

$$m(r \otimes v) = r \cdot v;$$

the tensor multiplication $O_K^\times : H(K) \to H(V, V \backslash K) \otimes H(K), O_M^\times : H(K) \to H(M) \otimes H(K)$ by

$$O_K^\times(v) = O_K \otimes v, \quad O_M^\times(v) = O_M \otimes v;$$

the projection $P : H(M^\times) \to H_n(M^\times)$ by

$$P(q) = q_n, \quad \text{if } q = \sum_k q_k, \ q_k \in H_k(M^\times).$$

Lemma 1.29. *(cf. Dold* [**19**, Lemma VII.6.14, p. 210]*) Suppose that V is an ANR. Then the maps*

$$\psi_0, \psi_1 : (V, V\backslash K) \times K \longrightarrow M^\times \times V \text{ given by}$$

$$\begin{array}{ll} \psi_0(v,k) = (v,k,v), & \\ \psi_1(v,k) = (v,k,k), & \qquad v \in V,\ k \in K, \end{array}$$

induce the same homomorphism in homology:

$$\psi_{0*} = \psi_{1*}.$$

Proof. Let

$$Q = \mathbf{I} \times D \cup \{0,1\} \times V \times K \subset \mathbf{I} \times V \times K,$$

where $\mathbf{I} = [0,1],\ D = \{(v,k) \in V \times K : v = k\}$ is the diagonal of $V \times K$. Note that D is closed since V is Hausdorff. Therefore Q is also closed. Consider a function $\alpha : Q \to V$ given by

$$\begin{array}{ll} \alpha(0,v,k) = v, & \\ \alpha(1,v,k) = k, & \\ \alpha(t,k,k) = k, & \qquad v \in V, k \in K, t \in \mathbf{I}. \end{array}$$

Clearly α is continuous. Then, since Q is a closed subset of $\mathbf{I} \times V \times K$ and V is an ANR, there is an extension of α to a neighborhood of Q. And since Q contains $\mathbf{I} \times D$, we assume that α is now defined on $\mathbf{I} \times W$, where W is an open neighborhood of D in $V \times K$. Suppose maps

$$\begin{array}{ll} \eta_i : (W, W\backslash D) \longrightarrow M^\times \times V, & \\ \varphi_i : (V \times K, (V \times K)\backslash D) \longrightarrow M^\times \times V, & \qquad i = 0,1, \end{array}$$

are given by the same formulas as ψ_i :

$$\begin{array}{l} \eta_0(v,k) = \varphi_0(v,k) = (v,k,v), \\ \eta_1(v,k) = \varphi_1(v,k) = (v,k,k). \end{array}$$

Then η_0 and η_1 are homotopic:

$$\eta_t(v,k) = (v,k,\alpha(t,v,k)), \quad 0 \le t \le 1.$$

Consider the following commutative diagram for $i = 0,1$:

$$\begin{array}{ccc} (V, V\backslash K) \times K & & \\ \downarrow^{j} & \overset{\psi_i}{\searrow} & \\ (V \times K, (V \times K)\backslash D) & \overset{\varphi_i}{\longrightarrow} & M^\times \times V. \\ \uparrow^{j'} & \overset{\eta_i}{\nearrow} & \\ (W, W\backslash D) & & \end{array}$$

where j, j' are inclusions. Since W is open and D is closed, j'_* is an isomorphism by excision. We also know that $\eta_0 \sim \eta_1$, so $\eta_{0*} = \eta_{1*}$. Therefore we have $\varphi_{0*} = \varphi_{1*}$. And since $\psi_i = \varphi_i j$, we finally conclude that $\psi_{0*} = \psi_{1*}$. □

Theorem 1.30 (Generalized Dold's Lemma). *(cf.* [**19**, Lemma VII.6.13, p. 210]*) Suppose that K is arcwise connected and the map $\Phi : H(K) \to H(V)$ is given as the composition of the following homomorphisms:*

$$\begin{array}{l} \Phi : H_i(K) \xrightarrow{O_K^\times} H_{n+i}((V, V\backslash K) \times K) \xrightarrow{(\delta \times Id)_*} H_{n+i}((V, V\backslash K) \times V \times K) \\ \xrightarrow{(Id \times t)_*} H_{n+i}((V, V\backslash K) \times K \times V) \xrightarrow{(I \times Id)_*} H_{n+i}(M^\times \times V) \\ \xrightarrow{P \otimes Id} H_n(M^\times) \otimes H_i(V) \xrightarrow{m} H_i(V). \end{array}$$

Then

$$\Phi = i_*.$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} & & H(M\times K) & \xrightarrow{\psi_*} & H(M\times K\times V) & & \\ & \nearrow^{O_M^\times} & \downarrow_{incl} & & \downarrow_{I_*\otimes Id} & & \\ H(K) & & H((M,M\backslash K)\times K) & \xrightarrow{\psi_*} & H(M^\times)\otimes H(V) & \xrightarrow{m(P\otimes Id)} & H(V), \\ & \searrow^{O_K^\times} & \uparrow^{incl} & \nearrow & & & \\ & & H((V,V\backslash K)\times K) & & & & \end{array}$$

where the vertical arrows are induced by inclusions and ψ is given by

$$\psi(v,k) = (v,k,k), \qquad v \in M,\ k \in K,$$

so $\psi = (I \times Id)(Id \times \delta)$. The diagram is commutative, because the left triangle commutes by Definition 6.5. Now, according to Lemma 7.1,

$$\psi_* : H((V,V\backslash K)\times K) \to H(M^\times \times V)$$

is also induced by

$$\psi'(v,k) = (v,k,v), \qquad v \in V,\ k \in K,$$

so $\psi' = (Id \times t)(\delta \times Id)$. Then the lower path defines Φ. Therefore so does the upper one. Hence

$$\Phi = m(P\otimes Id)\psi_* O_M^\times.$$

Consider $u \in H_i(K)$ and $w = \delta_*(u) \in (H(K)\otimes H(V))_i$. Then

$$w = \sum_{k+l=i} a_k \otimes b_l, \qquad a_k \in H_k(K),\ b_l \in H_l(V).$$

But $(\eta\otimes Id)\delta_*(u) = u$, where $\eta : H(K) \to \mathbf{Q}$ is the augmentation. Then

$$u = (\eta\otimes Id)\delta_*(u) = (\eta\otimes Id)w = (\eta\otimes Id)\sum_{k+l=i} a_k\otimes b_l = \eta(a_0)\otimes b_i.$$

Therefore, $b_i = i_*(u)$ and a_0 is represented by some $p \in K$. Since $O_M \in H_n(M)$, we have

$$\begin{aligned} \Phi(u) = m(P\otimes Id)\psi_*(O_M\otimes u) \quad &= m(PI_*\otimes Id)(O_M\otimes w) \\ &= m(PI_*\otimes Id)(O_M\otimes \textstyle\sum_{k+l=i}(a_k\otimes b_l)) \\ &= m(\textstyle\sum_{k+l=i} PI_*(O_M\otimes a_k)\otimes b_l) \\ &= m(I_*(O_M\otimes a_0)\otimes b_i) \\ &= I_*(O_M\otimes p)\cdot i_*(u). \end{aligned}$$

Finally, we observe that $l_p i_p(x) = I(x,p)$ for any $x \in M, \quad p \in K$. Therefore we have

$$\begin{aligned} \Phi(u) \quad &= l_{p*}i_{p*}(O_M)\cdot i_*(u) & \\ &= l_{p*}(s(p))\cdot i_*(u) & \text{by Proposition 6.4} \\ &= i_*(u) & \text{by Proposition 6.3.}\square \end{aligned}$$

In a fashion similar to the proof of Proposition 6.7 the original Dold's Lemma follows from this theorem.

1.8. The Lefschetz Number of a Pair. Now we recall some fundamentals of the theory of the Lefschetz number, see [**19**, p. 207-208] and [**24**, p. 19-20]. Let

$$E_q^* = Hom(E_{-q}), \quad E^* = \{E_q^*\},$$
$$(E^* \otimes E)_k = \bigotimes{}_{q+i=k}(E_q^* \otimes E_i), \quad E^* \otimes E = \{(E^* \otimes E)_k\}.$$

Now we define the following maps

$$e : (E^* \otimes E)_0 \longrightarrow \mathbf{Q} \text{ by } \quad e(u \otimes v) = u(v) \ (\textit{the evaluation map}),$$
$$\theta : (E^* \otimes E)_0 \longrightarrow Hom(E, E) \text{ by } \quad [\theta(a \otimes b)](u) = (-1)^{|b|\cdot|u|} a(u) \cdot b,$$

where $|w|$ stands for the degree of w.

Proposition 1.31. [**24**, Theorem II.1.5, p. 20] *If $h : E \to E$ is an endomorphism of degree zero of a finitely generated graded module, then we have*

$$e(\theta^{-1}(h)) = L(h).$$

What follows is an adaptation of Gorniewicz's argument [**24**, pp. 38-40] to the new situation. The next two lemmas are trivial.

Lemma 1.32. *Let $J : H(V, V \backslash K) \to (H(K))^*$ be a homomorphism of degree $(-n)$ given by*

$$J(u)(v) = I_{*n}(u \otimes v), \quad u \in H(V, V \backslash K), \ v \in H(K).$$

Then we have $I_{*n} = e(J \otimes Id)$, so that the following diagram commutes

$$\begin{array}{ccc} (H(V, V\backslash K) \otimes H(K))_n & \xrightarrow{J \otimes Id} & ((H(K))^* \otimes H(K))_0 \\ \downarrow^{I_*} & \swarrow_e & \\ H_n(M^\times) \simeq \mathbf{Q}. & & \end{array}$$

Lemma 1.33. *Let*

$$a = (J \otimes Id)(Id \otimes \varphi)\delta_*(O_K) = (J \otimes \varphi)\delta_*(O_K) \in ((H(K)^* \otimes H(K))_0.$$

Then we have

$$e(a) = I_*(Id \otimes \varphi)\delta_*(O_K).$$

Lemma 1.34. *Let $\varphi : H(V) \to H(K)$ be a homomorphism. Then the following diagram commutes:*

$$\begin{array}{ccc} H(V, V\backslash K) \otimes H(V) \otimes H(K) & \xrightarrow{J \otimes \varphi \otimes Id} & (H(K))^* \otimes H(K) \otimes H(K) \\ \downarrow^{Id \otimes t_*} & & \downarrow^{Id \otimes t_*} \\ H(V, V\backslash K) \otimes H(K) \otimes H(V) & \xrightarrow{J \otimes Id \otimes \varphi} & (H(K))^* \otimes H(K) \otimes H(K) \\ \downarrow^{PI_* \otimes Id} & & \downarrow^{e \otimes Id} \\ H_n(M^\times) \otimes H(V) & & \mathbf{Q} \otimes H(K) \\ \downarrow^{m} & & \downarrow^{m} \\ H(V) & \xrightarrow{\quad \varphi \quad} & H(K). \end{array}$$

Proof. The first square trivially commutes. For the second, consider $\to\downarrow$. Then we get:

$$\begin{aligned} m(e \otimes Id)(J \otimes Id \otimes \varphi) &= m(e(J \otimes Id) \otimes \varphi) \\ &= m(I_{*n} \otimes \varphi) && \text{by Lemma 8.2} \\ &= I_{*n} \cdot \varphi && \text{by definition of } m. \end{aligned}$$

For ↓→, we get:

$$\begin{aligned}\varphi m(PI_* \otimes Id) &= \varphi m(I_{*n} \otimes Id) && \text{by definition of } P\\ &= \varphi(I_{*n} \cdot Id) && \text{by definition of } m\\ &= I_{*n} \cdot \varphi && \text{by linearity of } \varphi.\square\end{aligned}$$

The following theorem implies Theorem 2.2.

THEOREM 1.35. *Suppose V is an ANR, K is an arcwise connected space, $H(K)$ is finitely generated. Then for any homomorphism $\varphi : H(V) \to H(K)$ we have*

$$L(\varphi i_*) = I_*(Id \otimes \varphi)\delta_*(O_K),$$

where $i : K \to V$ is the inclusion.

Proof. Let $a = \sum_i a_i \otimes a'_i$, where $a_i \in (H(K))^*, a'_i \in H(K)$. We start in the left upper corner of the above diagram with $\delta_*(O_K) \otimes u$, where $u \in H(K)$. Consoder ↓→. Then we get $\varphi i_*(u)$ by Theorem 7.2. For →↓, we get:

$$\begin{aligned}&m(e \otimes Id)(Id \otimes t_*)(J \otimes \varphi \otimes Id)(\delta_*(O_K) \otimes u)\\ &= m(e \otimes Id)(Id \otimes t_*)((J \otimes \varphi)\delta_*(O_K) \otimes u)\\ &= m(e \otimes Id)(Id \otimes t_*)(a \otimes u) && \text{by Lemma 8.3}\\ &= m(e \otimes Id)(Id \otimes t_*)(\textstyle\sum_i a_i \otimes a'_i \otimes u)\\ &= m(e \otimes Id)\textstyle\sum_i (-1)^{|a'_i|\cdot|u|}(a_i \otimes u \otimes a'_i)\\ &= \textstyle\sum_i m((-1)^{|a'_i|\cdot|u|}e(a_i \otimes u) \otimes a'_i)\\ &= \textstyle\sum_i m((-1)^{|a'_i|\cdot|u|}a_i(u) \otimes a'_i) && \text{by definition of } e\\ &= \textstyle\sum_i (-1)^{|a'_i|\cdot|u|}a_i(u) \cdot a'_i && \text{by definition of } m\\ &= \textstyle\sum_i \theta(a_i \otimes a'_i)(u) && \text{by definition of } \theta\\ &= \theta(a)(u).\end{aligned}$$

Thus $\theta(a) = \varphi i_* : H(K) \to H(K)$. Since $H(K)$ is a finitely generated graded module, Proposition 8.1 applies and we have

$$L(\varphi i_*) = L(\theta(a)) = e(a).$$

Now the statement follows from Lemma 8.3. □

1.9. A Lefschetz-Type Coincidence Theorem for Maps to an Open Subset of a Manifold.

Recall that if $(V, V\backslash K)$ is a manifold then according to Theorem 2.1 $\varphi = g_* f_!$ satisfies an identity that connects it to the coincidence index. Our next goal is to show that even if V is not a manifold under certain purely homological conditions we can construct φ satisfying that identity. This leads to the proof of a Lefschetz-type theorem for Case 2.

PROPOSITION 1.36. *Suppose $f : (X, X\backslash N) \to (V, V\backslash K)$, $g : X \to K$ are continuous maps, and there is a $\mu \in H_n(X, X\backslash N)$ satisfying:*

: *(a) $f_*(\mu) = O_K$,*

where $f_* : H_n(X, X\backslash N) \to H_n(V, V\backslash K)$, and there is a homomorphism $\varphi : H(V) \to H(K)$ of degree 0 satisfying:

: *(b) $\varphi f_* = g_*$,*

where $f_* : H(X) \to H(V)$.
Then the following holds

$$I_{fg} = I_*(Id \otimes \varphi)\delta_*(O_K),$$

i.e., I_{fg} is the image of O_K under the composition of the following maps:

$$H(V,V\backslash K)\xrightarrow{\delta_*}H(V,V\backslash K)\otimes H(V)\xrightarrow{Id\otimes\varphi}H(V,V\backslash K)\otimes H(K)\xrightarrow{I_*}H(M^\times).$$

Proof. (cf. Gorniewicz [**24**, pp. 15-16]) The following diagram commutes:

$$\begin{array}{ccccc}
H(V,V\backslash K)\otimes H(V) & \xleftarrow{Id\otimes f_*} & H(V,V\backslash K)\otimes H(X) & \xrightarrow{Id\otimes g_*} & H(V,V\backslash K)\otimes H(K) \\
\uparrow^{\alpha_1} & & \uparrow^{\alpha_2} & & \uparrow^{\alpha_3} \\
H((V,V\backslash K)\times V) & \xleftarrow{(Id\times f)_*} & H((V,V\backslash K)\times X) & \xrightarrow{(Id\times g)_*} & H((V,V\backslash K)\times K) \\
\uparrow^{\delta_*} & & \uparrow^{(f,Id)_*} & & \downarrow^{I_*} \\
H(V,V\backslash K) & \xleftarrow{f_*} & H(X,X\backslash N) & \xrightarrow{(f,g)_*} & H(M^\times),
\end{array}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3$ are the isomorphisms from the Künneth theorem. If we start with $\mu\in H(X,X\backslash N)$ in the middle of the lower row of the diagram, then from commutativity of the diagram it follows that

$$\begin{array}{lll}
I_*\alpha_3^{-1}(Id\otimes\varphi)\alpha_1\delta_*(O_K) & = I_*\alpha_3^{-1}(Id\otimes\varphi)\alpha_1\delta_*f_*(\mu) & \text{by (a)} \\
 & = I_*\alpha_3^{-1}(Id\otimes\varphi)(Id\otimes f_*)\alpha_2(f,Id)_*(\mu) & \text{the left half} \\
 & = I_*\alpha_3^{-1}(Id\otimes\varphi f_*)\alpha_2(f,Id)_*(\mu) & \\
 & = I_*\alpha_3^{-1}(Id\otimes g_*)\alpha_2(f,Id)_*(\mu) & \text{by (b)} \\
 & = I_{fg} & \text{the right half.}
\end{array}$$

□

It is obvious that both (a) and (b) are satisfied when f induces isomorphisms $H(X)\simeq H(V)$ and $H_n(X,X\backslash N)\simeq H_n(V,V\backslash K)$. Another example: if the maps $f,g:X\to M$ satisfy $f_{*n}\neq 0,\ g_*=0$, then we can select $\varphi=0$, so that $L(\varphi)=1$.

Now Proposition 9.1 and Theorem 8.5 imply the following.

Theorem 1.37 (Lefschetz-Type Coincidence Theorem). *Suppose X is a topological space, $N\subset X$, M is an oriented connected compact closed n-manifold, $V\subset M$ is an ANR, $K\subset V$ is an arcwise connected space, $H(K)$ is finitely generated, $(M,V,M\backslash K)$ is an excisive triad. Suppose maps*

$$f:(X,X\backslash N)\longrightarrow(V,V\backslash K),\quad g:X\longrightarrow K,$$

and a homomorphism $\varphi:H(V)\to H(K)$ satisfy the conditions of Proposition 9.1. Then the coincidence index is equal to the Lefschetz number:

$$I_{fg}=L(\varphi i_*).$$

Moreover, if $L(\varphi i_*)\neq 0$, then (f,g) has a coincidence.

Remark.: We proved the theorem for singular homology, but the proof is algebraic except for generalized Dold's Lemma 7.2. And since it holds for Čech homology, then so does the theorem.

The following two examples show limits of applicability of this result.

Example 1.38. (Dranishnikov [**20**, Lemma 1.9]) There is a multivalued u.s.c. retraction $\Phi:\mathbf{D}^n\to\mathbf{S}^{n-1}$:

$$\Phi(x)=\{y\in\mathbf{S}^{n-1}:|y-x|\geq 4|x|^2-3|x|\}.$$

Example 1.39. Let $\mathbf{M}^2$ be the Möbius band, given in cylindrical coordinates by: $z=\theta,\ -1\leq r\leq 1,\ 0\leq\theta\leq\pi$, with the top and bottom edges identified. Let $f:(\mathbf{M}^2,\partial\mathbf{M}^2)\to(\mathbf{D}^2,\mathbf{S}^1)$ be the projection on the horizontal plane and $g:\mathbf{M}^2\to\mathbf{S}^1$ the projection on the z-axis.

Observe that Φ in Example 9.3 has no fixed points, while in Example 9.4 g is homotopic to a map g' such that the pair (f, g') has no coincidence. This is reflected in the fact that $\Phi(x)$ fails to be acyclic for $|x| \leq 1/2$, while in Example 9.4 f_* does not satisfy condition (a).

1.10. The Generalized Lefschetz Number and Case 2. In Corollary 10.5 below we will see how one can use the Vietoris-Begle Theorem 4.4 to avoid the restriction on relative behavior of $f : (X, X\backslash N) \to (V, V\backslash K)$ and consider only $f : X \to V$. For this purpose, we would like to be able to deal with the Lefschetz number of $i_*\varphi_{fg} : H(V) \to H(V)$, instead of $\varphi_{fg}i_* : H(K) \to H(K)$, as before. Then, if $H(V)$ is not finitely generated, we need to define the generalized Lefschetz number $\Lambda(\cdot)$, as in [**24**, pp. 20-23].

Let $h : E \to E$ be an endomorphism of an arbitrary vector space E. Denote by $h^{(n)} : E \to E$ the nth iterate of h. Then the kernels

$$\ker h \subset \ker h^{(2)} \subset \ldots \subset \ker h^{(n)} \subset \ldots$$

form an increasing sequence of subspaces of E. Let

$$N(h) = \bigcup_n \ker h^{(n)} \text{ and } \tilde{E} = E/N(h).$$

Then h induces the endomorphism $\tilde{h} : \tilde{E} \to \tilde{E}$.

Definition 1.40. Let $h = \{h_q\}$ be an endomorphism of degree 0 of a graded vector space $E = \{E_q\}$ and suppose $\tilde{E}$ is finitely generated. Then the *generalized Lefschetz number (in the sense of Leray)* of h is given by

$$\Lambda(h) = \sum_q (-1)^q tr(\tilde{h}_q) = L(\tilde{h}).$$

Proposition 1.41. [**24**, II.2.3, p. 22] *If E is a finitely generated graded vector space then $\Lambda(h) = L(h)$.*

Proposition 1.42. [**24**, II.2.4, p. 22] *Let E, E' be graded modules and suppose that the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{k} & E' \\ \uparrow^{h} & \nwarrow^{l} & \uparrow^{h'} \\ E & \xrightarrow{k} & E'. \end{array}$$

If $\Lambda(h)$ is defined, then so is $\Lambda(h')$ and $\Lambda(h) = \Lambda(h')$.

Theorem 1.43. *Under conditions of Theorem 9.2, we have*

$$I_{fg} = \Lambda(i_*\varphi_{fg}).$$

Proof. From Proposition 10.3 and commutativity of the diagram we have:

$$\begin{array}{ccc} H(K) & \xrightarrow{i_*} & H(V) \\ \uparrow^{\varphi_{fg}i_*} & \nwarrow^{\varphi_{fg}} & \uparrow^{i_*\varphi_{fg}} \\ H(K) & \xrightarrow{i_*} & H(V). \end{array} \quad \square$$

Theorem 10.4 implies the Coincidence Theorem of Gorniewicz [**24**, p. 38], as follows:

Corollary 1.44. *Suppose X is a topological space, $V \subset \mathbf{R}^n$ is open. Suppose*

$$f, g : X \to V$$

are two continuous maps such that f is Vietoris and g is compact (i.e., $\overline{g(X)}$ is compact). If $\Lambda(g_* f_*^{-1}) \neq 0$ with respect to Čech homology over $\mathbf{Q}$, then the pair (f, g) has a coincidence.

Proof. Since g is a compact map, there is a finite connected polyhedron K such that $g(X) \subset K \subset V$. We can assume that $V \subset M = \mathbf{S}^n$. Then $(M, V, V \backslash K)$ is an excisive triad, V is an ANR, and K is arcwise connected space. Let $N = f^{-1}(K)$. Then we have $Coin(f, g) \subset N$. Since f_* is an isomorphism by Proposition 4.4, all the conditions of Proposition 9.1 are satisfied. Therefore by the theorem, we conclude that $I_{fg} = \Lambda(g_* f_*^{-1})$. □

See Gorniewicz [**24**, pp. 40-43] for applications of this theorem to the study of fixed points of multivalued maps on polyhedra, ANRs, etc.

2. Topological Convexity.

2.1. Introduction. The origin of our fixed point theorem is the following two classical results due to Kakutani-Ky Fan-Glicksberg [**42, 45, 23**] and Browder [**7**] respectively (see also [**72**]).

Theorem 2.1 (Kakutani Fixed-Point Theorem). *Let X be a nonempty convex compact subset of a locally convex Hausdorff topological vector space, and let $F : X \to X$ be an u.s.c. multifunction with nonempty convex closed images. Then F has a fixed point.*

Theorem 2.2 (Browder Fixed-Point Theorem). *Let X be a nonempty convex compact subset of a Hausdorff topological vector space, and let $G : X \to X$ be a multifunction with nonempty convex images and fibers relatively open in X. Then G has a fixed point.*

Similarly, our selection theorem unites the following two results due to Michael [**49**] (see [**60**] for further results) and Browder [**7**].

Theorem 2.3 (Michael Selection Theorem). *Let X be a paracompact Hausdorff topological space, and let Y be a Banach space. Let $T : X \to Y$ be a l.s.c. multifunction having nonempty closed convex images. Then T has a continuous selection.*

Theorem 2.4 (Browder Selection Theorem). *Let X be a paracompact Hausdorff topological space, and let Z be any topological vector space. Let $T : X \to Z$ be a multifunction having nonempty convex images and open fibers. Then T has a continuous selection.*

Our main goal is to provide a uniform approach to these four results. We present a theorem that contains as immediate corollaries both the Kakutani and Browder fixed point theorems for multivalued maps on topological vector spaces and a theorem containing both the Michael and Browder selection theorems, as well as a number of recent results [**2, 4, 15, 31, 38, 39, 56, 57, 66, 68**]. The initial motivation for this study was provided by McLinden [**48**].

Our approach is based on Michael's and Browder's techniques [**50, 7**] and the study of abstract convexity structures on topological spaces originated in works of Michael [**51**], Van de Vel [**67**], Horvath [**36**], and others. Given a topological (or uniform) space Y, Van de Vel introduces the class of "convex" sets as a class of subsets of Y closed under intersections. Horvath defines "convex hulls" of finite subsets of Y. Michael, on the other hand, considers an analogue of convex combination functions of vector spaces:

$$C(d_0, \ldots, d_n, a_0, \ldots, a_n) = \sum_{i=0}^{n} d_i a_i,$$

where $(d_0, \ldots, d_n)$ is an element of the n-simplex Δ_n, for certain combinations $(a_0, \ldots, a_n)$ of elements of the space. For topological vector spaces, convex combination functions are continuous with respect to $d_0, \ldots, d_n$. But for locally convex topological vector spaces, these functions are continuous with respect to $a_0, \ldots, a_n$ as well.

DEFINITION 2.5. (Michael [**51**]) Let Δ_n be the n-simplex spanned on the unit vectors of $\mathbf{R}^{n+1}$. Then a *convex structure* on a metric space (Y, ρ) is a sequence of pairs $\{(M_n, k_n)\}_{n=1}^{\infty}$, where M_n is a subset of Y^n and $k_n : M_n \times \Delta_{n-1} \to Y$ is a map such that the following conditions hold:

(a) if $x \in M_1$, then $k_1(x, 1) = x$,

(b) if $x \in M_n, n \geq 2, i \leq n$, then $\partial_i x \in M_{n-1}$, and if $t_i = 0$ for $t \in \Delta_{n-1}$, then $k_n(x, t) = k_{n-1}(\partial_i x, \partial_i t)$, where ∂_i is the operator that omits the ith coordinate,

(c) if $x \in M_n$ and $x_i = x_{i+1}$, then for $t \in \Delta_{n-1}$, we have

$$k_n(x, t) = k_{n-1}(\partial_i x, t_1, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_n),$$

(d) for fixed $x \in M_n$, the map $k_n(x, \cdot)$ is continuous,

(e) for every $\varepsilon > 0$ there exists a neighborhood V of the diagonal in $Y \times Y$ such that, for all $n \in \mathbf{N}$ and all $x, y \in M_n$, if $(x_i, y_i) \in V, 1 \leq i \leq n$, then $\rho(k_n(x, t), k_n(y, t)) < \varepsilon$ for all $t \in \Delta_{n-1}$.

We follow Michael's approach because, as we will see, it is especially convenient for selection and fixed-point problems. For the former, Y does not have to be "convex". Following Michael, we avoid this situation, while in Van de Vel's and Horvath's constructions there is the largest "convex" set (see Sections 17-19). Our construction extends Michael's definition in such a way that most of Van de Vel's and Horvath's fixed-point and selection results are included.

We relax Michael's conditions in several ways. First, we do not assume that Y is metrizable but only uniform. Second, we allow the convex combination function to be multivalued. Third, instead of a sequence of maps $\{k_n\}_{n=1}^{\infty}$ connected by conditions (b) and (c), we use a sole multifunction C from a subset of the set $\Delta(Y)$ of all formal convex combinations of elements of Y into Y, which makes it easier to prove existence of C. We further generalize this construction by introducing a collection of "approximative" convexity multifunctions $\{C_V\}$ satisfying certain continuity conditions (D) and (E) similar to (d) and (e) above. Condition (E) allows us to carry out most of the selection and fixed point constructions and only as the last step do we consider various continuity conditions with respect to t (condition (D)), which ensures continuity of selections and existence of fixed points of continuous maps. The conditions of Michael's definition do not hold for non-locally convex topological vector spaces and we have to deal with them in order to obtain Theorems 11.2 and 11.4. To resolve this problem, we introduce a second topology Z on Y. As a result, the convexity satisfies the two continuity requirements above, but with respect to two (possibly different) topologies. Consequently, by choosing an appropriate topological structure on Y, we are able to obtain Theorems 11.1 (for $Z = Y$) and 11.2 (for Y discrete) as immediate corollaries of our fixed point result (Theorem 26.3). In the same manner we derive Theorems 11.3 and 11.4 from our selection theorem (Theorem 23.4).

The chapter is organized as follows. In Section 12, we collect necessary preliminaries from point-set topology. In Section 13, for a given uniform space Y, we introduce a convexity on Y as a family of multifunctions from $\Delta(Y)$ into Y. In Sections 16-19, we consider convexity functions of locally convex and non-locally convex topological vector spaces, H-spaces/c-spaces, H-spaces of generalized Zima type/l.c.-spaces, G-convex spaces, as well as convex structures of Van de Vel and Michael. In Section 20, we show that large classes of topological spaces can be equipped with a convex structure. In Section 22, we present some definitions and supplementary results from general topology. In Chapter 3 we prove our selection and fixed point theorems.

2.1.1. *Notation and Preliminaries.* Let $F : X \to Y$ be a multifunction (a set-valued map $F : X \to 2^Y$), where X, Y are topological spaces. Defining a multifunction is equivalent to defining its graph:

$$Graph(F) = \bigcup \{\{x\} \times F(x) : x \in X\} \subset X \times Y.$$

Any set $G \subset X \times Y$ determines a multifunction, as follows:

$$F(x) = \{y \in Y : (x, y) \in G\}.$$

Then for any subset A of X, we let

$$F(A) = \bigcup\nolimits_{x \in A} F(x),$$

and for any subset B of Y, we let

$$F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

In particular, *fibers* of F are

$$F^{-1}(y) = \{x \in X : y \in F(x)\},\ y \in Y.$$

We call F *upper-semicontinuous* (u.s.c.) if $F^{-1}(B)$ is closed for any closed B, and *lower-semicontinuous* (l.s.c.) if $F^{-1}(B)$ is open for any open B. If $G : Y \to Z$ is another multifunction, then the *composition* of F and G is the multifunction $G \circ F : X \to Z$ given by

$$G \circ F(x) = \bigcup_{y \in F(x)} G(y)$$

We say that $x \in X$ is a *fixed point* of $F : X \to X$ if $x \in F(x)$. We say that a single-valued function $g : X \to Y$ is a *selection* of $F : X \to Y$ if $g(x) \in F(x)$ for all $x \in X$.

A topological space is called *acyclic* if its reduced Čech homology groups over the rationals are trivial. The set of all acyclic spaces includes convex subsets of topological vector spaces and contractible spaces. An u.s.c. map with compact acyclic images is called an *acyclic multifunction*.

A multivalued map $F : X \to Y$ is called *admissible in the sense of Gorniewicz* [**26**] if it is closed valued u.s.c. and there exist a topological space Z and two single-valued continuous maps $p : Z \to X,\ q : Z \to Y$ such that p is proper (i.e., $f^{-1}(B)$

is compact for any compact $B \subset Y$) and for any $x \in X$,

$$\begin{array}{l} (i) \quad p^{-1}(x) \text{ is acyclic, and} \\ (ii)\ q(p^{-1}(x)) \subset F(x). \end{array}$$

A *uniform base* $\mathcal{B}$ for a uniform space Y [**22**] is a family of subsets of $Y \times Y$ containing the diagonal $B_0 = \{(z,z) : z \in Y\}$ with two operations:

$$\begin{array}{ll} -V & = \{(x,y) : (y,x) \in V\}, \\ V + W & = \{(x,z) : \text{ there exists such a } y \in Y \text{ that } (x,y) \in V,\ (y,z) \in W\}, \\ & \text{where } V,\ W \in \mathcal{B}, \end{array}$$

so that the following conditions are satisfied:

(U1) if $V, W \in \mathcal{B}$ then there exists a $U \in \mathcal{B}$ such that $U \subset V \cap W$,
(U2) if $V \in \mathcal{B}$ then there exists a $W \in \mathcal{B}$ such that $2W \subset V$,
(U3) $-V = V$ for any $V \in \mathcal{B}$.

The notation $|x-y| < V,\ V \in \mathcal{B}$, is used when $(x,y) \in V$, and we let

$$\begin{array}{ll} B(z_0, V) & = \{z \in Y : |z_0 - z| < V\}, \\ B(A, V) & = \bigcup_{z \in A} B(z, V), \end{array}$$

where $z_0 \in Y,\ A \subset Y$.

We will need the following partition of unity result.

Theorem 2.6 (Michael's Lemma). [**22**, Theorem 5.1.9, p. 301] *If X is a normal space and $\gamma = \{Q_i : i \in J\}$ is a locally finite open cover of X then there exists a partition of unity subordinate to γ, i.e., there are continuous functions $f_k : X \to [0,1],\ k \in J$, satisfying*

$$\begin{array}{c} f_k(x) = 0 \textit{ for any } x \notin Q_k,\ k \in J, \textit{ and} \\ \sum_{k \in J} f_k(x) = 1 \textit{ for any } x \in X. \end{array}$$

2.2. Convexity on Uniform Spaces. Throughout the paper we fix an infinite cardinal number ω and an index set I with $|I| = \omega$. We assume that ω is large enough in the sense that $\omega \geq 2^{|X|}$ for each space X involved. Let Δ_ω be the infinite dimensional simplex spanned by the unit vectors $e_k,\ k \in I$, of $[0,1]^\omega$, i.e.,

$$\Delta_\omega = \{(d_i)_{i \in I} \in [0,1]^\omega : \text{only finite number of } d_i \text{ are not zero, } \sum_{i \in I} d_i = 1\}.$$

For any nonempty subset K of I, let Δ_K denote the convex hull of the set $\{e_k : k \in K\}$ in Δ_ω :

$$\Delta_K = \{d = (d_i)_{i \in I} \in \Delta_\omega : i \notin K \Rightarrow d_i = 0\}$$

and let Δ_n be any n-simplex in Δ_ω spanned by some unit vectors. We let

$$\Delta(Y) = \Delta_\omega \times Y^\omega,$$

and we assume the following:

Convention.: If $(d,a) \in \Delta(Y)$ then $d_i \in [0,1],\ a_i \in Y,\ i \in I$, are the coordinates of $d \in [0,1]^\omega,\ a \in Y^\omega$.

For any $A \subset Y$, we define the set of *all formal convex combinations* of elements of A :

$$\Delta(A) = \{(d,a) \in \Delta(Y) : d_i \neq 0 \Rightarrow a_i \in A,\ i \in I\}.$$

The following are also fixed:

(c1) Y is a uniform space with a minimal (uniform) open base $\mathcal{B}$ (i.e., one with the smallest cardinality), partially ordered by inclusion,

(c2) Z is a topological space on Y (the topology of Z is not necessarily the uniform topology of Y),

(c3) $\mathcal{V}$ is a class of multifunctions that will be specified later,

(c4) $\mathcal{A} \subset 2^Y \backslash \{\emptyset\}$ is a class of subsets of Y called *admissible sets* ($\mathcal{A}$ may be empty),

(c5) $conv : \mathcal{A} \to 2^Y \backslash \{\emptyset\}$ is a function, and $conv(A)$ is called the *convex hull* of $A \in \mathcal{A}$,

(c6) $\mathcal{C} = \{A \in \mathcal{A} : conv(A) \subset A\} \cup \{\emptyset\}$ is the set of *convex subsets* of Y,

(c7) $Q = \bigcup_{A \in \mathcal{A}} \Delta(A)$,

(c8) Q' is a subset of $\Delta(Y)$ containing $\bigcup_{A \in \mathcal{A}, W \in \mathcal{B}} \Delta(B(A, W))$,

(c9) $C_V : Q' \to Y$, $V \in \mathcal{B}$, are multifunctions called *(approximative) convex combinations.*

Definition 2.7 (Main Definition). The triple $\kappa = (Y, \{C_V\}, Z)$ is called a *convexity* associated with $\mathcal{A}$, *conv*, $\mathcal{V}$, Q' (this part will often be suppressed) if the following conditions are satisfied:

(D): for any $V \in \mathcal{B}$, $a \in Y^\omega$, if $\Delta_n \times \{a\} \subset Q'$, $n \geq 0$, then the multifunction $C_V(\cdot, a) : \Delta_n \to Z$ belongs to $\mathcal{V}$,

(E): for any $U \in \mathcal{B}$, there exist $V, W \in \mathcal{B}$ such that

$$C_V(\Delta(B(A, W))) \subset B(conv(A), U) \text{ for all admissible } A \subset Y.$$

If $Q' = \Delta(Y)$ then the convexity is called *full.* When $C_V = C$ for all $V \in \mathcal{B}$, we denote the convexity by (Y, C, Z). Then condition (E) is equivalent to

(E0): for any $U \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that

$$C(\Delta(B(A, W))) \subset B(conv(A), U) \text{ for all admissible } A \subset Y.$$

Thus as a set, Y is equipped with two topologies (in applications they either coincide or one of them is discrete), which can be illustrated by the following diagram:

$$\begin{array}{ccc} Q' & \xrightarrow{C_V} & Y \\ \cup & & \| \\ \Delta_n \times \{a\} & \xrightarrow{C_V(\cdot, a)} & Z, \end{array}$$

where the vertical lines $\|$ indicate that Y and Z are defined on the same set.

Proposition 2.8. *If (Y, C, Z) is a convexity then the following condition is satisfied:*

(γ) $C(\Delta(A)) \subset \overline{conv}(A)$*, the closure of* $conv(A)$ *in* Y*, for all admissible* $A \subset Y$.

Proof. By $(E0)$, for any $U \in \mathcal{B}$ there is a $W \in \mathcal{B}$ such that

$$C(\Delta(A)) \subset C(\Delta(B(A, W))) \subset B(conv(A), U) \text{ for all admissible } A \subset Y.$$

The statement then follows. □

2.3. An Alternative Definition of Convexity. An alternative way to introduce a convexity is given in the following proposition. Here, instead of fixing $\mathcal{A}$ and *conv* as in (c4) and (c5) and then obtaining $\mathcal{C}$ as in (c6), we start with $\mathcal{C}$.

Proposition 2.9. *Suppose* $Y \in \mathcal{C} \subset 2^Y$ *and* $\mathcal{C}$ *satisfies*

(c6′) $\mathcal{D} \subset \mathcal{C} \Rightarrow \cap \mathcal{D} \in \mathcal{C}$.

We define $\mathcal{A} \subset 2^Y$ *and* $conv : \mathcal{A} \to 2^Y \backslash \{\emptyset\}$ *as follows:*

(c4′) $\mathcal{A} = \{A \in 2^Y \backslash \{\emptyset\} : \textit{there is a } D \in \mathcal{C} \textit{ such that } A \subset D\} = 2^Y \backslash \{\emptyset\}$,

(c5′) $conv(A) = \cap\{D \in \mathcal{C} : A \subset D\},\ A \in \mathcal{A}$.

Then we have

(c6) $\mathcal{C} = \{A \in \mathcal{A} : conv(A) \subset A\}$.

Proof. Suppose $A \in \mathcal{A}$ and $conv(A) \subset A$. Then by (c5′), we have $\bigcap\{D \in \mathcal{C} : A \subset D\} \subset A$, hence $\bigcap\{D \in \mathcal{C} : A \subset D\} = A$. Then by (c6′), we have $A \in \mathcal{C}$. Thus we have proved that $\mathcal{C} \supset \{A \in \mathcal{A} : conv(A) \subset A\}$. To prove the identity, take $A \in \mathcal{C}$. Then trivially A belongs to $\mathcal{A}$ and $conv(A) = A$. Therefore (c6) holds. □

Suppose the sets Q, Q' and multifunctions $C_V, V \in \mathcal{B}$, are fixed. Then Proposition 14.1 implies that if we start with $\mathcal{C}$ (convex sets) satisfying (c6′) instead of $\mathcal{A}$ (admissible sets) and *conv* (the convex hull) given in (c4) and (c5), we arrive at a convexity in the sense of (c4)-(c6) and Definition 13.1. Throughout the paper we assume that we are using the original definition of convexity (via $\mathcal{A}$ and *conv*), unless we say that $\kappa = (Y, \{C_V\}, Z)$ is a *convexity associated with* $\mathcal{C}$ (and $\mathcal{V},\ Q'$). Then we are using the above construction, with $\mathcal{A}$ and *conv* derived from $\mathcal{C}$.

The following properties immediately follow from the definition.

Proposition 2.10. *Suppose $\kappa = (Y, C, Z)$ is a convexity associated with $\mathcal{C} \subset 2^Y$. Then we have the following*

(1) for any $A \in \mathcal{A},\ conv(A) \in \mathcal{C}$,

(2) for any $A \in \mathcal{A},\ A \subset conv(A)$,

(3) for any $D \in \mathcal{C},\ conv(D) = D$.

Next we consider two alternative ways to present condition $(E0)$.

Proposition 2.11. *Suppose $\mathcal{C} \subset 2^Y$ and $Y \in \mathcal{C}$, and $\kappa = (Y, C, Z)$ satisfies conditions of Proposition 14.1. Then the following condition is equivalent to* $(E0)$:

$(E1)$ *for any $U \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that*

$$C(\Delta(B(D, W))) \subset B(D, U) \textit{ for any convex } D \subset Y.$$

Proof. $(E1) \Rightarrow (E0)$: If $A \in \mathcal{A}$ then we have $A \subset D = conv(A) \in \mathcal{C}$ by (1) and (2) of Proposition 14.2. Therefore by $(E1)$, for any $U \in \mathcal{B}$, there exists a $W \in \mathcal{B}$ such that

$$C(\Delta(B(A, W))) \subset C(\Delta(B(D, W))) \subset B(D, U) = B(conv(A), U).$$

$(E0) \Rightarrow (E1)$: Since $\mathcal{C} \subset \mathcal{A}$, from $(E0)$ it follows that for any $U \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that

$$C(\Delta(B(D, W))) \subset B(conv(D), U) = B(D, U)$$

(by (3) of Proposition 14.2). □

Proposition 2.12. *Suppose $\mathcal{C} \subset 2^Y$ and $Y \in \mathcal{C}$, and $\kappa = (Y, C, Z)$ satisfies conditions of Proposition 14.1 and condition (γ) of Proposition 13.2 is satisfied. Then the condition*

$(E2)$ *for any $U \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that*

$$conv(B(D, W)) \subset B(D, U) \textit{ for any convex } D \subset Y.$$

implies condition $(E0)$.

Proof. First, notice that $Y \in \mathcal{C}$ implies that $\mathcal{A} = 2^Y \backslash \{\emptyset\}$, so (γ) holds for all subsets of Y. Let $U' \in \mathcal{B}$. Then there is a $U \in \mathcal{B}$ with $2U \subset U'$. Suppose A is an admissible set. Then $A \subset D = conv(A) \in \mathcal{C}$ by (1) and (2) of Proposition 14.2. By

$(E2)$, there is a $W \in \mathcal{B}$ such that $conv(B(D,W)) \subset B(D,U')$. Therefore by (γ), we have

$$C(\Delta(B(A,W))) \subset C(\Delta(B(D,W))) \subset \overline{conv}(B(D,W)) \subset \overline{B(D,U')} \subset B(D,U),$$

so $(E0)$ holds. $\square$

2.4. A Strong Convexity. Let $F_i : X \to Y,\ i \in J$, be multifunctions, where J is a directed set. Then we say that $\{F_i : i \in J\}$ *converges uniformly on* $N \subset X$ to a multifunction $F : N \to Y$ if for any $U \in \mathcal{B}$, there exists an $i_0 \in A$ such that

$$F_i(x) \subset B(F(x), U) \text{ for all } x \in N,\ i \in J,\ i > i_0.$$

Let Ω denote the set of all finite subsets of the index set I. For a fixed $d \in \Delta_\omega$, we define elements of the product uniformity of $\{d\} \times Y^\omega$ as follows: for any $W \in \mathcal{B}$, $m \in \Omega$, we let

$$\begin{aligned}
&W^m = \{((d,a),(d,a')) \in \Delta(Y) \times \Delta(Y) : (a_j, a'_j) \in W,\ j \in m\},\\
&B^*((d,a), W^m) = \{(d,a') \in \Delta(Y) : (a_j, a'_j) \in W,\ j \in m\},\\
&B^*(S, W^m) = \textstyle\bigcup_{s \in S} B^*(s, W^m),
\end{aligned}$$

where $(d,a) \in \Delta(Y),\ S \subset \Delta(Y)$.

Consider the following conditions on the objects defined in (c1)-(c9) that loosely correspond to conditions (a)-(e) of Michael's Definition 11.5:

(α) (convergence) $\{C_V : V \in \mathcal{B}\}$ converges uniformly on Q to a multifunction $C : Q \to Y$,

(β) (permutations) if $(d,a),(d',a') \in Q$ and $\sum_{a_i = y} d_i = \sum_{a'_i = y} d'_i$ for any $y \in Y$, then $C(d,a) = C(d',a')$,

(γ) (convex hull) $C(\Delta(A)) \subset \overline{conv}(A)$, the closure in Y, for all admissible $A \subset Y$,

$(\delta) = (D)$ (d-continuity) for any $V \in \mathcal{B}$, $a \in Y^\omega$, if $\Delta_n \times \{a\} \subset Q',\ n \geq 0$, then the multifunction $C_V(\cdot, a) : \Delta_n \to Z$ belongs to $\mathcal{V}$,

(ε) (a-continuity) for any $V \in \mathcal{B}$ and any $U \in \mathcal{B}$, there exist $W \in \mathcal{B},\ m \in \Omega$, such that

$$C_V(B^*((d,a), W^m)) \subset B(C_V(d,a), U) \text{ for all } (d,a) \in Q.$$

Remark: Condition (ε) may be understood as if the family $\{C_V(d,\cdot) : d \in \Delta_\omega\}$ is "*equi-uniformly u.s.c.*", although, unless C_V is compact valued, it does not have to be u.s.c..

Definition 2.13. The triple $\kappa = (Y, \{C_V\}, Z)$ is called a *strong convexity* if conditions (α), (β), (γ), (δ) and (ε) are satisfied.

As a direct consequence of the definitions above, we obtain the following for a strong convexity.

Lemma 2.14. *For any $A \subset Y,\ W \in \mathcal{B},\ m \in \Omega$, we have*

$$\Delta(B(A,W)) \subset B^*(\Delta(A), W^m).$$

Proof.

$$
\begin{aligned}
&B^*(\Delta(A), W^m) = \\
&= \{(d,a) \in \Delta(Y) : \text{ there is a } (d,a') \in \Delta(A) \text{ such that } (a_j, a'_j) \in W,\ j \in m\} \\
&= \{(d,a) \in \Delta(Y) : \text{ there is an } a' \in Y \text{ such that } d_i \neq 0 \Rightarrow a'_i \in A \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and } j \in m \Rightarrow (a_j, a'_j) \in W\} \\
&= \{(d,a) \in \Delta(Y) : \ j \in m, d_j \neq 0 \Rightarrow \text{ there exists an } a'_j \in A \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{with } a_j \in B(a'_j, W)\} \\
&\supset \{(d,a) \in \Delta(Y) : \text{ for any } j \in I, d_j \neq 0 \Rightarrow\ a_j \in B(A,W)\} \\
&= \Delta(B(A,W)). \ \square
\end{aligned}
$$

THEOREM 2.15. *Conditions* (α), (γ) *and* (ε) *imply condition* (E)*, so any strong convexity is a convexity.*

Proof. Let $U \in \mathcal{B}$ be fixed, and let $U' \in \mathcal{B}$ satisfy $4U' \subset U$. By (α), there exists a $V_0 \in \mathcal{B}$ such that for all $V \subset V_0,\ V \in \mathcal{B}$, we have

$$C_V(d,a) \subset B(C(d,a), U') \text{ for all } (d,a) \in Q.$$

Fix such a V. By (ε), there exist $W \in \mathcal{B},\ m \in \Omega$, such that

$$C_V(B^*((d,a), W^m)) \subset B(C_V(d,a)), U') \text{ for all } (d,a) \in Q.$$

Combining these two inclusions, we obtain the following: there exist $V, W \in \mathcal{B},\ m \in \Omega$, satisfying

$$C_V(B^*((d,a), W^m)) \subset B(C_V(d,a), U') \subset B(C(d,a), 2U') \text{ for all } (d,a) \in Q.$$

If A is an admissible set then $\Delta(A) \subset Q$, so this inclusion holds for all $(d,a) \in \Delta(A)$. Hence

$$C_V(B^*(\Delta(A), W^m)) \subset B(C(\Delta(A)), 2U').$$

Applying consecutively Lemma 15.2, the above inclusion and condition (γ), we obtain

$$
\begin{aligned}
C_V(\Delta(B(A,W))) \subset C_V(B^*(\Delta(A), W^m)) \quad &\subset B(C(\Delta(A)), 2U') \\
&\subset B(\overline{conv}(A), 2U') \\
&\subset B(conv(A), U),
\end{aligned}
$$

so condition (E) is satisfied. $\square$

2.5. Convexity of Topological Vector Spaces.

DEFINITION 2.16. Let $\mathcal{V}$ be the class of all single-valued continuous maps. Then we say that the convexity is *continuous*.

Some examples of spaces with continuous convexity are listed in this and following sections.

DEFINITION 2.17. A continuous convexity $\kappa = (Y, \{C_V\}, Y)$ (here the topological structures of Y and Z coincide) associated with $\mathcal{A}$, *conv*, is called *simplicial* if for any $y \in Y$, we have $\{y\} \in \mathcal{A}$ and $conv(\{y\}) = \{y\}$.

PROPOSITION 2.18. *Let* Y *be a convex subset of a locally convex topological vector space. Then* (Y, C, Y) *with* C *given by*

$$C(d,a) = \sum_i d_i a_i, \quad (d,a) \in \Delta(Y),$$

is a simplicial (strong) full convexity associated with $\mathcal{A} = 2^Y$, $conv(A) = co(A),\ A \subset Y$ *(where* $co(A)$ *is the usual convex hull in a vector space).*

Proof. As a locally convex topological vector space Y has a convex base $\mathcal{B}$, i.e., $B(0,U)$ is convex for all $U \in \mathcal{B}$. Conditions (α) and (β) of Section 15 are trivially satisfied. To prove condition (γ) we notice that for any $A \subset Y$ we have

$$\begin{aligned} conv(A) &= co(A) \\ &= \bigcup_{n=0}^{\infty} \{\textstyle\sum_{i=0}^{n} t_i b_i : 1 \geq t_i \geq 0,\ b_i \in A,\ i = 0, \dots, n,\ \sum_{i=0}^{n} t_i = 1\} \\ &= C(\Delta(A)). \end{aligned}$$

Suppose $\mathcal{V}$ is the set of all single-valued continuous maps. But $C(\cdot, a) : \Delta_n \to Z$ is continuous as a linear map on a finite-dimensional space, so condition (δ) is satisfied for $Z = Y$. Let $U \in \mathcal{B}$, $(d,a) \in \Delta(Y)$ with $d_i = 0$ for $i > n$. Consider $(d', a') \in B^*((d,a), U^m)$, where $m = \{i : d_i \neq 0\} \in \Omega$. Therefore we have

$$\begin{aligned} d'_i = d_i,\ a'_i = a_i + c_i, &\text{ for all } i \in I, \text{ and} \\ c_i \in B(0,U) &\text{ for all } i \in m. \end{aligned}$$

Then it follows that

$$\begin{aligned} C(d', a') &= \textstyle\sum_{i \in m} d'_i a'_i = \sum_{i \in m} d_i (a_i + c_i) \\ &= \textstyle\sum_{i \in m} d_i a_i + \sum_{i \in m} d_i c_i \\ &\in C(d,a) + \textstyle\sum_{i \in m} d_i B(0,U) \\ &\subset C(d,a) + B(0,U) = B(C(d,a), U). \end{aligned}$$

Thus condition (ε) is satisfied. □

A subset A of a topological vector space E is called *locally convex* [**30**] if it has a base of convex neighborhoods in the relative uniformity. Then the above construction applies to A without change. It is also known that if E has a separating dual (i.e., for any $x \in E \backslash \{0\}$ there exists a continuous linear functional l such that $l(x) \neq 0$), then every compact convex subset of E is locally convex. Another generalization of local convexity is given below.

Definition 2.19. (cf. [**30**]) A subset A of a topological vector space is called *admissible in the sense of Klee* if for all compact subsets K of A and for all neighborhoods V of the origin, there is a continuous map $h_V : K \to A$ such that

(1) $span(h_V(K))$ is finite dimensional, and

(2) $y - h_V(y) \in V$ for all $y \in K$.

It is known that all convex subsets of a locally convex space are admissible in the sense of Klee. We can also show that a compact convex subset, admissible in the sense of Klee, has a convexity. In fact, a more general statement is true.

Proposition 2.20. *Suppose Y is Hausdorff. For each $V \in \mathcal{B}$, suppose also that*

(1) Y_V is a subset of Y,

(2) $\kappa_V = (Y_V, K_V, Y_V)$ is a strong full simplicial convexity associated with

(a) $\mathcal{A}_V \supset \{\{y\} : y \in Y_V\}$,

(b) $conv_V$ such that $conv_V(\{y\}) = \{y\},\ y \in Y_V$,

(c) $Q'_V = \Delta(Y_V)$,

(3) $h_V : Y \to Y_V$ is a uniformly continuous function satisfying $|y - h_V(y)| < V$ for all $y \in Y$.

Then there is a (strong) full simplicial convexity $\kappa = (Y, \{C_V\}, Y)$ associated with

(a) $\mathcal{A} = \{\{y\} : y \in Y\}$,

(b) $conv$ such that $conv(\{y\}) = \{y\},\ y \in Y$,

(c) $Q' = \Delta(Y)$,

where

$$C_V(d,a) = K_V(d,\tilde{h}_V(a)),\ V \in \mathcal{B},\ (d,a) \in \Delta(Y),$$

and $\tilde{h}_V : Y^\omega \to Y^\omega$ *is given by* $\tilde{h}_V(a) = (h_V(a_i))_{i\in I},\ a \in Y^\omega$.

Proof. We have a family of convexities $\kappa_V = (Y_V, K_V, Y_V),\ V \in \mathcal{B}$, i.e., with $K_V : Q'_V \to Y_V$. From the formulas above we construct a new convexity $\kappa = (Y, \{C_V\}_{V\in\mathcal{B}}, Y)$, i.e., with $C_V : Q' \to Y,\ V \in \mathcal{B}$. Suppose $\mathcal{V}$ is the set of all single-valued continuous maps. We have

$$Q = \{(d,a) \in \Delta(Y) :\ \text{for some } y \in Y,\ d_i \neq 0 \Rightarrow a_i = y\},$$
$$C(d,a) = y \text{ if } (d,a) \in Q \text{ and } d_i \neq 0 \Rightarrow a_i = y.$$

Then (β) and (γ) are trivially satisfied. Condition (δ) (d-continuity) for $\{C_V\}$ follows directly from condition (δ) for $K_V,\ V \in \mathcal{B}$.

Next we prove condition (ε) (a-continuity) for $\{C_V\}$. Fix $V, U \in \mathcal{B}$. Then by (ε) for κ_V, there are $W \in \mathcal{B}$ and $m \in \Omega$ such that

$$K_V(B^*((d,b),W^m) \cap \Delta(Y_V)) \subset B(K_V(d,b),U) \text{ for all } (d,b) \in Q_V, \tag{2.1}$$

where Q_V is defined for $\kappa_V = (Y_V, K_V, Y_V)$ as in (c7) Section 13. Fix these W and m. Since $\kappa_V = (Y_V, K_V, Y_V)$ is simplicial, $\{y\}$ is an element of $\mathcal{A}_V$ for any $y \in Y_V$, so

$$Q_V \supset Q_V^0 = \{(d,b') \in \Delta(Y) :\ \text{for some } y \in Y_V,\ d_i \neq 0 \Rightarrow b'_i = y\}.$$

Therefore (16.1) and the definition of $B^*((d,b),W^m)$ imply the following. Suppose $(d,b) \in \Delta(Y_V)$. Suppose also that (a) $(d,b) \in Q_V^0$, and (b) $|(d,b) - (d,b')| < W^m$. Then we have

$$K_V(d,b') \in B(K_V(d,b),U). \tag{2.2}$$

Now from uniform continuity of $h_V : Y \to Y_V$, it follows that there is an $E \in \mathcal{B}$ such that for any $q, q' \in Y$,

$$|q - q'| < E \Rightarrow |h_V(q) - h_V(q')| < W. \tag{2.3}$$

Take arbitrary $(d,a) \in Q$ and $(d,a') \in B^*((d,a),E^m)$. Then, first, there is a $y \in Y$ such that $d_i \neq 0 \Rightarrow a_i = y$, so $d_i \neq 0 \Rightarrow h_V(a_i) = h_V(y)$. It follows that

$$(d,\tilde{h}_V(a)) \in Q_V^0,$$

(so we have (a)), and second, by (16.3), we have

$$|(d,\tilde{h}_V(a)) - (d,\tilde{h}_V(a'))| < W^m$$

(so we have (b)). Therefore (16.2) holds for $b = \tilde{h}_V(a), b' = \tilde{h}_V(a')$:

$$K_V(d,\tilde{h}_V(a')) \in B(K_V(d,\tilde{h}_V(a)),U), \text{or}$$
$$C_V(d,a') \in B(C_V(d,a),U).$$

Therefore, we have

$$C_V(B^*((d,a),E^m)) \subset B(C_V(d,a),U),$$

so condition (ε) holds for $\kappa = (Y, \{C_V\}, Y)$.

To prove (α), we consider $(d,a) \in Q$. Then, for some $y \in Y$, $d_i \neq 0$ implies $a_i = y$. Hence we have

$$\begin{array}{lll} C_V(d,a) & = K_V(d, \tilde{h}_V(a)) & \text{by definition of } C_V \\ & \in K_V(\Delta(h_V(y))) & \text{by definition of } \Delta(\cdot) \\ & \subset \overline{conv_V}(h_V(\{y\})) & \text{by condition } (\gamma) \text{ for } \kappa_V \\ & = \overline{\{h_V(y)\}} & \text{because } \kappa_V \text{ is simplicial} \\ & = \{h_V(y)\} & \text{because } Y_V \subset Y \text{ is Hausdorff.} \end{array}$$

Therefore by the hypothesis, for any $V \in \mathcal{B}$, there is a $U = V \in \mathcal{B}$ such that

$$|C(d,a) - C_V(d,a)| \quad = |y - h_V(y)| \ < V.$$

This implies the uniform convergence of $\{C_V\}$ to C. $\square$

Corollary 2.21. *Let K be a compact convex subset of a topological vector space. If K is admissible in the sense of Klee then K has a full simplicial convexity.*

Proof. By definition, we have $h_V : K \to K$ satisfying (1) and (2). Therefore $span(h_V(K))$ is a finite dimensional and, therefore, locally-convex topological vector space. So by Proposition 16.3, we have $Y_V = span(h_V(K)) \cap K$ has a strong simplicial convexity. Now the statement follows from the above proposition. $\square$

If the topological vector space is not locally convex the proof of Proposition 16.3 does not work, which motivates the next definition.

Definition 2.22. A continuous convexity (Y, C, Z) is called *discrete* if Y is discrete (then condition (E) turns into $C(\Delta(A)) \subset conv(A)$ for all $A \in \mathcal{A}$).

Proposition 2.23. *Let Z be a convex subset of a topological vector space. Then (Y, C, Z) (Y is discrete) with C given by*

$$C(d,a) = \sum_i d_i a_i, \quad (d,a) \in \Delta(Y),$$

is a discrete full convexity associated with $\mathcal{A} = 2^Y$, $conv(A) = co(A)$, $A \subset Y$.

Proof. Condition $(E0)$ is trivially satisfied, because the uniform base $\mathcal{B}$ of Y consists of only one element $B_0 = \{(b,b) : b \in Y\}$. Now we observe that $C(\cdot, a) : \Delta_n \to Z$ is continuous as a linear map on a finite-dimensional space, so (D) of Definition 13.1 holds for $\mathcal{V}$ the class of all continuous maps. $\square$

2.6. Horvath spaces. The following notion, originating from the work of Horvath [**35, 36**], is a generalization of the convex hull in a topological vector space.

Definition 2.24. A pair $(Z, \{\Gamma_A\})$ will be called a *Horvath space*, if Z is a topological space and $\{\Gamma_A\}$ is a family of contractible subsets of Z indexed by all finite subsets of Z so that

$$\Gamma_A \subset \Gamma_B \text{ whenever } A \subset B.$$

($(Z, \{\Gamma_A\})$ is called an *H-space* [**2**], or a *c-space* [**36**]). A set $A \subset Z$ is called *H-convex* if $\Gamma_D \subset A$ for any finite $D \subset A$, and the *H-convex hull* of a set $A \subset Y$ is given by

$$conv^*(A) = \bigcup \{\Gamma_D : D \subset A, \ D \text{ is finite}\}.$$

The proof of the proposition below follows the proof of Theorem 1 of Horvath [**35**].

Proposition 2.25. *Let $(Z, \{\Gamma_A\})$ be a Horvath space. Then there is a full discrete convexity (Y, C, Z) (i.e., Y is discrete) associated with some $C : \Delta(Y) \to Y$, $\mathcal{A} = 2^Y \setminus \{\emptyset\}$, and conv given by: $conv(A) = conv^*(A),\ A \in \mathcal{A}$.*

Proof. Let $\mathcal{C}$ be the set of all H-convex sets. Then $\mathcal{C}$ satisfies the conditions of Proposition 14.1, so all we need is to construct $C : \Delta(Y) \to Z$ satisfying conditions (D) and $(E0)$. We inductively construct $C : \Delta(Y) \to Z$. Let $o(d, a)$ denote the number of nonzero d_i, and for $n \geq 0$, let

$$\Delta_n(Y) = \{(d, a) \in \Delta(Y) : o(d, a) \leq n + 1\}.$$

First, we define C on $\Delta_0(Y)$ as follows. If $(d, a) \in \Delta_0(Y)$ then there is an i such that $d_i = 1$ and $a_i = b \in Y$. Select an $x \in \Gamma_{\{b\}}$ and let $C(d, a) = x$. Then, for any $K = \{k\} \subset I$, the map $C(\cdot, a) : \Delta_K \to Z$ is continuous. Now, assume inductively that for $n \geq 0,\ C$ is defined on $\Delta_n(Y)$, and for any $K \subset I$ with $\mid K \mid \leq n + 1$ and any $a \in Y^\omega$, the following holds:

(1) the map $C(\cdot, a) : \Delta_K \to Z$ is continuous,

(2) $C(\Delta_K, a) \subset \Gamma_A$, where $A = \{a_k : k \in K\}$.

Let $a \in Y^\omega,\ M \subset I,\ \mid M \mid = n + 2$, and $B = \{a_i : i \in M\}$. By assumption, $C(\cdot, a)$ is defined on the boundary of Δ_M. For any $K \subset M$ with $\mid K \mid \leq n + 1,\ \Delta_K$ is a subset of the boundary, so we have $C(\Delta_K, a) \subset \Gamma_A$, where $A = \{a_i : i \in K\} \subset B$. Therefore by definition of a Horvath space, we have $C(\Delta_K, a) \subset \Gamma_B$. But Γ_B is contractible, so we can extend $C(\cdot, a)$ to Δ_M so that $C(\Delta_M, a) \subset \Gamma_B$. Hence C is defined on $\Delta_{n+1}(Y)$ and conditions (1), (2) hold. Thus, by induction, condition (D) is satisfied for $\mathcal{V}$ the class of all continuous maps.

Next, (2) implies that for any finite $D \subset Y$, we have $C(\Delta(D)) \subset \Gamma_D$. Therefore we have

$$(\gamma')\ C(\Delta(A)) \subset conv^*(A), \text{ for any } A \subset Y,$$

which implies condition $(E0)$ of Definition 13.1 since the uniform base $\mathcal{B}$ of Y is discrete. □

The next definition provides an analogue of local convexity for Horvath spaces that have a certain uniform base.

Definition 2.26. (1) (Horvath [**36**]) We say that a Horvath space $(Y, \{\Gamma_A\})$ is an *l.c.-space* if there is a uniform base $\mathcal{B}$ such as for any $U \in \mathcal{B}$,

$$B(A, U) \text{ is an H-convex set whenever } A \subset Y \text{ is H-convex.}$$

A metric space (Y, d) is called a *metric l.c. space* if it is a c-space and $\forall \varepsilon > 0,\ \{y \in Y : d(y, E) < \varepsilon\}$ is an H-convex set if E is an H-convex set, and open balls are H-convex.

(2) (Hadžić [**31**]) We say that a Horvath space $(Y, \{\Gamma_A\})$ is of *generalized Zima type* if there is a uniform base $\mathcal{B}$ such that for every $U \in \mathcal{B}$, there exists a $V \in \mathcal{B}$ such that for every finite subset D of Y and every H-convex subset A of Y the following holds:

$$A \cap B(z, V) \neq \emptyset \text{ for every } z \in D \Longrightarrow A \cap B(u, U) \neq \emptyset \text{ for every } u \in \Gamma_D.$$

Proposition 2.27. *Suppose that*

(1) $(Y, \{\Gamma_A\})$ is an l.c.-space, or

(2) $(Y, \{\Gamma_A\})$ is a Horvath space of generalized Zima type with H-convex points. Then there is a full simplicial convexity (Y, C, Y) associated with $\mathcal{C} = \{$H-convex sets$\}$.

Proof. In the proof of Proposition 17.2 we constructed a function $C : \Delta(Y) \to Z$ (Y is discrete) satisfying conditions (γ') and (D). But these conditions do not depend on the uniformity of Y. Therefore if $Z = Y$, we immediately obtain condition (D) for $C : \Delta(Y) \to Y$. Notice that in both cases points are H-convex (for (1) it follows from H-convexity of balls). Now all we need to prove is condition $(E0)$.

(1): In Section 14, *conv* is defined as follows:

$$(c5')conv(A) = \cap\{D \in \mathcal{C} : A \subset D\},\ A \in \mathcal{A}.$$

Therefore, we have

$$conv^*(A) \subset conv(A),\ A \subset Y.$$

Then from (γ') we obtain the following:

$$(\gamma'')\ C(\Delta(A)) \subset conv(A),\ \text{for any } A \subset Y,$$

Now, by condition (γ'') and (3) of Proposition 14.2, we have

$$C(\Delta(B(A,W))) \subset conv(B(A,W)) \subset B(A,W),$$

and we obtain condition $(E1)$ of Proposition 14.3.

(2): The definition implies that for any $A \in \mathcal{C}$, any finite D, we have

$$D \subset B(A,V) \Longrightarrow \Gamma_D \subset B(A,U),$$

hence

$$conv^*(B(A,V)) \subset B(A,U).$$

Therefore by (γ'), we have

$$C(\Delta(B(A,V))) \subset B(A,U),$$

and we obtain condition $(E1)$ of Proposition 14.3. □

According to Horvath [**36**], so-called convex metric spaces [**65**] and hyperconvex spaces [**1**] are l.c.-spaces with $conv(\{y\}) = \{y\}$.

Proposition 2.28. [**36**, Corollary 4.3] *If $(Y, \{\Gamma_A\})$ is a complete metric l.c.-space with H-convex points then any H-convex set is an AR.*

2.7. Van de Vel Uniform Convex Structures. The following definitions are due to Van de Vel [**68, 67**], but see also [**71, 70, 62**].

Definition 2.29. [**67**, pp. 3 and 304] A pair $(Y, \mathcal{C})$, where $\mathcal{C}$ is a family of subsets of Y, called *V-convex sets*, is called a *Van de Vel uniform convex structure* if

(1) The empty set $\emptyset$ and the universal set Y are in $\mathcal{C}$,

(2) $\mathcal{C}$ is stable for intersections, that is, if $\mathcal{D} \subset \mathcal{C}$ is nonempty, then $\cap\mathcal{D}$ is in $\mathcal{C}$ (same as (c6′) of Proposition 14.1),

(3) $\mathcal{C}$ is stable for nested unions, that is, if $\mathcal{D} \subset \mathcal{C}$ is nonempty and totally ordered by inclusion, then $\cup\mathcal{D}$ is in $\mathcal{C}$,

(4) there is a uniform base $\mathcal{B}$ such that for each $U \in \mathcal{B}$, there is a $V \in \mathcal{B}$ such that

$$\text{for any } A \in \mathcal{C},\ conv(B(A,V)) \subset B(A,U),$$

where the *V-convex hull conv* is defined as in (c5′) of Proposition 14.1:

$$conv(A) = \bigcap\{D \in \mathcal{C} : A \subset D\},\ A \subset Y$$

(then by (c4′), $\mathcal{A} = 2^Y$).

Proposition 2.30. *If $(Y,\mathcal{C})$ is a Van de Vel uniform convex structure such that all elements of $\mathcal{C}$ are AR's, then there is a full continuous convexity (Y,C,Y) associated with $\mathcal{C}$.*

Proof. For each finite $D\subset Y$, let $\Gamma_D=\bigcap\{C\in\mathcal{C}:D\subset C\}\in\mathcal{C}$. Then we have a Horvath space $(Y,\{\Gamma_D\})$. In the proof of Proposition 17.2 we constructed a function $C:\Delta(Y)\to Z$ (Y discrete) satisfying condition (D). This condition does not depend on the uniformity of Y. Therefore if $Z=Y$ we have condition (D) for $C:\Delta(Y)\to Y$. Hence all we need to prove is condition ($E0$). But condition (4) of the above definition is exactly condition ($E2$) of Proposition 14.4. □

Van de Vel says that his convex structure satisfies the S_4-*axiom* if two disjoint V-convex sets can be separated by two V-convex sets complement to each other. He calls V-convex hulls of finite sets *polytopes*. The statement below follows from Van de Vel's selection theorem.

Proposition 2.31. [**67**, Theorem 3.17, p. 446] *Let $(Y,\mathcal{C})$ be a metrizable Van de Vel uniform convex structure satisfying the S_4-axiom with compact polytopes such that $\mathcal{C}$ contains only connected sets. Then each non-empty V-convex subset of Y is an AR.*

Then these propositions imply the following.

Proposition 2.32. *Let $(Y,\mathcal{C})$ be a metrizable Van de Vel uniform convex structure satisfying the S_4-axiom with compact polytopes such that $\mathcal{C}$ contains only connected sets. Then there is a full continuous convexity (Y,C,Y) associated with $\mathcal{C}$.*

Thus we have shown that the object of Van de Vel's study is the same as ours.

2.8. Michael Convex Structures. As before, Δ_n stands for the n-simplex, and if $x\in\Delta_n$ then the coordinates of x are $x_0,...,x_n$. Assuming that $\mathcal{B}$ is a base of the metric uniformity and that Y is compact, we restate Definition 11.5 in a more convenient form (we interchange variables of k_n), which is equivalent to the Michael's definition if Y is compact.

Definition 2.33. A sequence of pairs $\{(M_n,k_n)\}$ is a *Michael convex structure* if for all $n\geq 0,\ M_n\subset Y^{n+1},\ k_n:\Delta_n\times M_n\to Y$ (M_n can be empty) and the following are satisfied

(a) if $x\in M_0$, then $k_0(1,x)=x$,

(b) if $x\in M_n,\ n\geq 1,\ i\leq n$, then $\partial_i x\in M_{n-1}$, and if $t_i=0$ for $t\in\Delta_n$, then $k_n(t,x)=k_{n-1}(\partial_i t,\partial_i x)$, where ∂_i is the operator that omits the i-th coordinate,

(c) if $x\in M_n,\ n>i\geq 0$, and $x_i=x_{i+1}$, then for $t\in\Delta_n$

$$k_n(t,x)=k_{n-1}(t_1,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_n,\partial_i x),$$

(d) for each $x\in M_n$, the map $k_n(\cdot,x)$ is continuous,

(e) for any $U\in\mathcal{B}$ there is a $W\in\mathcal{B}$ such that for all $n\geq 0,\ t\in\Delta_n,\ x=(x_0,\dots,x_n),\ y=(y_0,\dots,y_n)\in M_n$, we have

$$(x_i,y_i)\in W,\ 0\leq i\leq n,\Longrightarrow (k_n(t,x),k_n(t,y))\in U.$$

A set $N\subset Y$ is called *M-admissible* if $N^n\subset M_{n-1}$ for all $n\geq 1$, and the *M-convex hull* of N is

$$conv^*(N)=\{k_{n-1}(t,x):x\in N^n,\ t\in\Delta_{n-1},\ n\geq 1\}.$$

The set N is called *M-convex* if it is M-admissible and for $n=1,2,...,\ k_{n-1}(\Delta_{n-1}\times N^n)\subset N$, i.e., if $conv^*(N)=N$.

Proposition 2.34. *Let $\{(M_n, k_n)\}$ be a Michael convex structure. Let*

$$\mathcal{A} = \{A \subset Y : B(A,W)^{n+1} \subset M_n \text{ for all } n \geq 0,\ W \in \mathcal{B}\},$$
$$conv(A) = \{k_n(t,x) : x \in A^{n+1},\ t \in \Delta_n,\ n \geq 0\},\ A \in \mathcal{A}$$

(notice that $\mathcal{A}$ can be empty even if M_n are not). Then (Y, C, Y) is a continuous convexity if $C : Q' \to Y$, where $Q' = \bigcup_{A\in\mathcal{A},W\in\mathcal{B}} \Delta(B(A,W))$ (cf. (c8)), is defined as follows: for any $(d,a) \in Q'$,

$$C(d,a) = k_n(t,x),$$

where

$$t = (d_{i_0}, \ldots, d_{i_n}) \text{ and } x = (a_{i_0}, \ldots, a_{i_n}),$$
$$\{i_k : k = 0, \ldots, n\} = \{i \in I : d_i \neq 0\},\ i_0 < \ldots < i_n,$$

(we assume that the index set I is totally ordered).

Proof. Suppose $A \in \mathcal{A}$. Then we have $B(A,W)^{n+1} \subset M_n$ for any $n \geq 0$ and $W \in \mathcal{B}$. Therefore k_n is defined on $\Delta_n \times B(A,W)^{n+1}$. Let $U \in \mathcal{B}$. Then by (e), there is a $W \in \mathcal{B}$ such that for all $A \in \mathcal{A},\ n \geq 0$, such that

$$t \in \Delta_n,\ x_0, \ldots, x_n \in A,\ y = (y_0, \ldots, y_n) \in M_n \text{ with } y_i \in B(x_i, W),\ 0 \leq i \leq n,$$

we have

$$k_n(t,y) \in B(k_n(t,x), U).$$

Or, for all $n \geq 0,\ y = (y_0, \ldots, y_n) \in M_n$ with $y_i \in B(A,W),\ 0 \leq i \leq n$, we have

$$\begin{aligned} k_n(\Delta_n \times \{y\}) &\subset B(k_n(\Delta_n \times A^{n+1}), U) \\ &\subset B(conv(A), U). \end{aligned}$$

Thus, for any $U \in \mathcal{B}$, there is a $W \in \mathcal{B}$ such that for all $A \in \mathcal{A},\ n \geq 0$, the following holds

$$k_n(\Delta_n \times B(A,W)^{n+1}) \subset B(conv(A), U). \tag{2.4}$$

Suppose now that $A \in \mathcal{A},\ W \in \mathcal{B}$, and $(d,a) \in \Delta(B(A,W))$. Then, $d_i \neq 0$ implies $a_i \in B(A,W)$. But we know that $C(d,a) = k_n(t,x)$, with x, t, n defined as in the statement of the proposition. Then we have $x_i \in B(A,W),\ i = 0, \ldots, n$. Hence

$$C(d,a) \in k_n(\Delta_n \times B(A,W)^{n+1}),$$

which, together with (19.1), gives condition $(E0)$.

To prove condition (D), we first notice that condition (b) implies that we can define C as follows. For any n large enough, any $(d,a) \in Q'$, let

$$C(d,a) = k_n(t,x),$$

where

$$t = (d_{i_0}, \ldots, d_{i_n}) \text{ and } x = (a_{i_0}, \ldots, a_{i_n}),$$
$$\{i_k : k = 0, \ldots, n\} \supset \{i \in I : d_i \neq 0\},\ i_0 < \ldots < i_n,$$

i.e., some of $d_{i_0}, \ldots, d_{i_n}$ can be zero. Therefore, if $(d,a), (d',a') \in Q'$ then there is an $n \geq 0$ such that $C(d,a) = k_n(t,x)$ and $C(d',a') = k_n(t',x')$. Now condition (D) follows from (d). □

The two definitions below are intended to generalize Michael's construction but, in fact, are limited to the "convex" situation, as are Horvath's and Van de Vel's.

Definition 2.35. (Pasicki [**58**], see also Bielawski [**5**]) A pair (E,Q) is a *weed* in a topological space Y if E is a (nonempty) set, $Q=\{Q_n\}_{n\in\mathbf{N}}$ ($\mathbf{N}=\{0,1,2,...\}$) is a sequence of mappings $Q_n:\Delta_n\times E^{n+1}\to Y$ satisfying

(1) if $t_i=0$ then $Q_n(t,e)=Q_{n-1}(\partial_i t,\partial_i e),$ where $e\in E^{n+1},\ i=0,\dots,n,$

(2) $Q_n(\cdot,e):\Delta_n\to Y$ is continuous for each $e\in E^n,\ n\in\mathbf{N}$.

Here all Michael's conditions are dropped except for (b) and (d). Then to prove his selection theorem, Pasicki [**58**, Definition 7.8 and Theorem 7.11, p. 90] adds another condition similar to ($E0$). It is clear that, unlike the one in the above definition, Michael convex structure can be empty. But more significant is the fact that, instead of sets $M_n\subset Y^{n+1}$, powers E^n of the set E are considered, which implies existence of the largest "convex" set $Y'=\cup_{n=1}^{\infty}Q_n(\Delta_{n-1},E^n)$. Therefore a weed is not a generalization of Michael convex structure as claimed by Pasicki [**58**, p. 91]. This is easy to see from the following example of a Michael convex structure that is not a weed: $Y=M_0=\{0,1\}\subset\mathbf{R},\ M_n=\emptyset$ for $n\geq 1,\ k_0(t,x)=x$ for all $x\in Y,t\in[0,1]$. Let $E=Y=\{0,1\},\ Q_0(t,x)=x$ for all $x\in Y,\ t\in[0,1]$. Then it is obvious that Q_1 does not exist.

More generally, an m-sphere $\mathbf{S}^m,\ m\geq 0$, can be equipped with a Michael convex structure as follows. Let $m=1$ and $Y=\mathbf{S}^1=\{z\in\mathbf{C}:|z|=1\}$. Let

$$\begin{aligned}
&R_n=\{(z_0,...,z_n)\in(\mathbf{S}^1)^{n+1}:\ \text{for all } k=0,...,n,\ Re(z_k)>0\},\\
&L_n=\{(z_0,...,z_n)\in(\mathbf{S}^1)^{n+1}:\ \text{for all } k=0,...,n,\ Re(z_k)<0\},\\
&T_n=\{(z_0,...,z_n)\in(\mathbf{S}^1)^{n+1}:\ \text{for all } k=0,...,n,\ Im(z_k)>0\},\\
&B_n=\{(z_0,...,z_n)\in(\mathbf{S}^1)^{n+1}:\ \text{for all } k=0,...,n,\ Im(z_k)<0\},\\
&M_n=R_n\cup L_n\cup T_n\cup B_n.
\end{aligned}$$

Then we can define

$$k_n(t_0,...,t_n,e^{is_0},...,e^{is_n})=\exp(i\sum_{k=0}^{k=n}t_k s_k),$$

where $(t_0,...,t_n)\in\Delta_n,\ (e^{is_0},...,e^{is_n})\in M_n$.

The next definition is an attempt to unite Michael's and Horvath's convex structures.

Definition 2.36. (Park and Kim [**57**]) A *generalized convex space* or a *G-convex space* is a triple (X,B,Γ), where X is a topological space, B is a nonempty subset of X, and for any nonempty finite subset D of B, $\Gamma(D)$ is a nonempty subset of X such that

(1) if $D\subset D'$ are finite subsets of B then $\Gamma(D)\subset\Gamma(D')$,

(2) for each finite $D\subset B$, there is a continuous map $\varphi_D:\Delta_n\to\Gamma(D)$, where $\mid D\mid=n+1$, such that for any nonempty $J\subset D$,

$$\varphi_D(\Delta_J)\subset\Gamma(J),$$

where Δ_J denotes the face of Δ_n corresponding to J (here it is assumed that there is a 1-1 correspondence between the elements of D and the vertices of Δ_n).

G-convex spaces are not generalizations of Michael convex structures (they do generalize Horvath spaces) as can be inferred from the diagram in [**57**, p. 176]; it follows from the fact that the set X is "convex". On the other hand, the uniform equicontinuity condition (e) is absent, which leads to the following consequences. Even though a Browder-type fixed point theorem can be proved for G-convex spaces, it is apparently impossible to derive the Kakutani Fixed Point Theorem 11.1 without additional assumptions.

2.9. Constructing a Convexity on a Topological Space. Throughout this section we assume that the index set I is well ordered.

Our goal is to construct a simplicial convexity on a given uniform space. There are a number of ways to construct multifunctions C_V (e.g., constant functions). But to apply it to fixed point problems (Corollary 27.1) we need a full simplicial convexity, i.e., C_V is defined on the whole $\Delta(Y)$ and

$$conv(\{y\}) = \{y\} \text{ for all } y \in Y,$$

or, equivalently, "all points are convex". We define a convexity C as follows. Let

$$\mathcal{A} = \{\{y\} : y \in Y\} \text{ and}$$
$$Q = \bigcup_{y\in Y} \Delta(\{y\}) = \{(d,a) \in \Delta(Y) : \text{for some } y \in Y, d_i \neq 0 \Rightarrow a_i = y\}$$

Recall that the base of $\Delta(Y)$ is given by the sets:

$$W^{\varepsilon,m} = \{((d,a),(d',a')) \in \Delta(Y) \times \Delta(Y) : \mid d_i - d'_i \mid < \varepsilon,\ (a_i,a'_i) \in W,\ i \in m\},$$
$$B^*((d,a),W^{\varepsilon,m}) = \{(d',a') \in \Delta(Y) : ((d,a),(d',a')) \in W^{\varepsilon,m}\},$$
$$B^*(S,W^{\varepsilon,m}) = \bigcup_{s\in S} B^*(s,W^{\varepsilon,m}),$$

for any $(d,a) \in \Delta(Y), S \subset Y, W \in \mathcal{B}$, $\varepsilon > 0,\ m \in \Omega$, where Ω is the set of all finite subsets of I.

Lemma 2.37. *Suppose Y is Hausdorff compact and infinite. Let $\Delta'(Y)$ be the quotient space of $\Delta(Y)$ with respect to the following equivalence relation:*

$$(d,a) \sim (d',a') \text{ if for any } y \in Y,\ \sum_{a_i=y} d_i = \sum_{a'_i=y} d'_i.$$

Let $C : Q \to Y$ be given by

$$C(x) = y \text{ for any } x \in \Delta(\{y\}).$$

Then

(i) $\Delta'(Y)$ is a Hausdorff uniform space and the quotient map $p : \Delta(Y) \to \Delta'(Y)$ is uniformly continuous,

(ii) (a) In Q, equivalence classes are $\Delta(\{y\}),\ y \in Y$, (b) $p(\Delta(\{y\}) = y,\ y \in Y$, and (c) $p(Q) = Y$.

Proof. First, $\Delta(Y)$ is normal, so $\Delta'(Y)$ is Hausdorff. Next, for any $(d,a) \in \Delta(Y),\ y \in Y$, let $S_y(d,a) = \sum_{a_i=y} d_i$. Then the set

$$R = \{((d,a),(d',a')) : \text{for any } y \in Y,\quad S_y(d,a) = S_y(d',a')\}$$

determines the equivalence relation $\sim$. To ensure that $\Delta'(Y)$ has a uniformity, we need to check that for any $W \in \mathcal{B}$, any $\varepsilon > 0,\ m \in \Omega$, there are $V \in \mathcal{B},\ \delta > 0,\ n \in \Omega$, such that

$$V^{\delta,n} + R + V^{\delta,n} \subset R + W^{\varepsilon,m} + R,$$

which means that the equivalence relation $\sim$ and the uniformity of $\Delta(Y)$ are weakly compatible (see [**40**, p. 24]), so (i) holds. We will show that $\Delta(Y) \times \Delta(Y) \subset$

$R+W^{\varepsilon,m}+R$. Consider

$$\begin{aligned} R+W^{\varepsilon,m}+R &= \{((d,a),(d',a')) : \text{there is an } (r,b)\in\Delta(Y) \text{ with} \\ &\qquad ((d,a),(r,b))\in R,\ ((r,b),(d',a'))\in W^{\varepsilon,m}+R\} \\ &= \{((d,a),(d',a')) : \\ &\quad \text{there is an } (r,b)\in\Delta(Y) \text{ with } S_y(d,a)=S_y(r,b), \forall y\in Y, \\ &\quad \text{there is an } (r',b')\in\Delta(Y) \text{ with } S_y(d',a')=S_y(r',b'), \forall y\in Y, \\ &\quad \text{and } \mid r_i-r'_i\mid<\varepsilon,\ (b_i,b'_i)\in W,\ i\in m\} \\ &= \{((d,a),(d',a')) : \text{there are } (r,b),(r',b')\in\Delta(Y) \text{ with} \\ &\qquad (1)\ \mid r_i-r'_i\mid<\varepsilon,\ (b_i,b'_i)\in W,\ i\in m, \\ &\qquad (2)\ S_y(d,a)=S_y(r,b), \forall y\in Y, \\ &\qquad (3)\ S_y(d',a')=S_y(r',b'), \forall y\in Y\}. \end{aligned}$$

Suppose $((d,a),(d',a'))\in\Delta(Y)\times\Delta(Y)$. Assume for simplicity that $m=\{0,1,...,M\}$. Choose $z\in Y$ such that $d_i\neq 0\Rightarrow z\neq a_i$ and $d'_i\neq 0\Rightarrow z\neq a'_i$ for all $i\in I$. To get $(r,b),(r',b')\in\Delta(Y)$ as above, let

$$r_i=r'_i=0,\ b_i=b'_i=z, \qquad i\in m=\{0,1,\ldots,M\},$$

Then condition (1) above is satisfied. Now, let

$$r_{M+i+1}=d_i,\ r'_{M+i+1}=d'_i,\ b_{M+i+1}=a_i,\ b'_{M+i+1}=a'_i,\ i=0,1,\ldots$$

(it means that we obtain (r,b) and (r',b') by "shifting" coordinates of (d,a) and (d',a') M steps to the right). Then conditions (2) and (3) above are also satisfied. Therefore we have

$$R+W^{\varepsilon,m}+R=\Delta(Y)\times\Delta(Y).$$

If $(d,a)\in Q$ and $d_i\neq 0$ then $a_i=y$ for all i and some $y\in Y$, and $p(d,a)=y$. Then for any $V\in\mathcal{B},\ n\in\Omega,\ \varepsilon>0$, we have

$$\begin{aligned} B^*((d,a),V^{\varepsilon,n})\cap Q &= \{(d',a')\in Q :\mid d_i-d'_i\mid<\varepsilon,\ (a_i,a'_i)\in V,\ i\in n\} \\ &= \{(d',a')\in\Delta(Y) : \text{for some } \ y'\in B(y,V), \\ &\qquad d'_i\neq 0\Rightarrow a'_i=y', \text{ and } \mid d_i-d'_i\mid<\varepsilon,\ i\in n\} \\ &= \textstyle\bigcup_{y'\in B(y,V)}\{(d',a')\in\Delta(Y) : d'_i\neq 0\Rightarrow a'_i=y', \\ &\qquad\qquad \mid d_i-d'_i\mid<\varepsilon,\ i\in n\}. \end{aligned}$$

Therefore we have

$$p(B^*((d,a),V^{\varepsilon,n})\cap Q)=B(y,V),$$

and we conclude that $p(Q)=Y$. $\square$

DEFINITION 2.38. Two functions $f:L\to Y$, $g:M\to Y$ are called *V-near on $K\subset L\cap M$*, where $V\in\mathcal{B}$, if $|f(x)-g(x)|<V$ for all $x\in K$. A uniform space Y is called an *approximative extension space for compact spaces*, or *AES(compact)* (cf. [**4**]), if for any Hausdorff compact space X, any $V\in\mathcal{B}$, any closed subset K of X, and any continuous function $f_0:K\to Y$, there exists a continuous function $f:X\to Y$ such that f_0 and f are V-near on K. And we call Y an *approximative neighborhood extension space for compact spaces, or $ANES$(compact)*, if for any Hausdorff compact X, any $V\in\mathcal{B}$, any closed subset K of X, and any continuous function $f_0:K\to Y$, there exists a neighborhood N of K and a continuous function $f:N\to Y$ such that f_0 and f are V-near on K.

In [**27**] *$ANES$(compact)* means that f_0 and f are not just V-near but V-homotopic.

First we construct a convexity for a space $Y\in$*$ANES$(compact)*.

LEMMA 2.39. *Let $Y \in ANES(compact)$ be Hausdorff compact. Then for any $V \in \mathcal{B}$, there exists a multifunction $C'_V : \Delta'(Y) \to Y$ such that*

(i) C'_V is uniformly u.s.c.,

(ii) C'_V is single-valued on some neighborhood N_V of $p(Q)$,

(iii) $C'_V(d,a) = Y$ for any $(d,a) \in \Delta'(Y) \backslash N_V$,

(iv) C'_V and C' are V-near on $p(Q)$.

Proof. By Lemma 20.1 $\Delta'(Y)$ is Hausdorff and $p(Q) = Y$ is compact. Hence $p(Q)$ is closed in $\Delta'(Y)$. Then according to the definition of $ANES(compact)$, for any $V \in \mathcal{B}$, there exists an open neighborhood $N_V \subset \Delta'(Y)$ of $p(Q)$ and a continuous function $C'_V : \overline{N_V} \to Y$ such that Id_Y and C'_V are V-near on $Y = p(Q)$. Therefore by the Cantor Theorem, C'_V is uniformly continuous (since $\overline{N_V}$ is compact). Now if we extend C'_V on the whole $\Delta'(Y)$ by putting $C'_V(d,a) = Y$ for $(d,a) \notin N_V$, then C'_V is uniformly u.s.c. (since N_V is open). □

THEOREM 2.40. *Let $Y \in ANES(compact)$ be Hausdorff compact and let $\mathcal{V}$ be the set of all u.s.c. multifunctions with values either singletons or Y. Then there exists a (strong) full convexity $\kappa = (Y, \{C_V\}, Y)$ associated with $\mathcal{A} = \{\{y\} : y \in Y\}$, $conv(\{y\}) = \{y\}$ for all $y \in Y$.*

Proof. By Lemma 20.3, for any $V \in \mathcal{B}$, we have a multifunction $C'_V : \Delta'(Y) \to Y$ satisfying (i)-(iv). Let $C_V = C'_V \circ p$. Then C_V is uniformly u.s.c. on $\Delta(Y)$ since p is uniformly continuous. Therefore conditions (ε) and (δ) of Definition 13.1 are satisfied. Also, according to (ii) and (iii), $C_V(\cdot, a)$ is a singleton or Y, so C_V belongs to $\mathcal{V}$. Now, (α) follows from (iv), (β) from the definition of the equivalence relation $\sim$, (γ) from the definition of C. □

If $\mathcal{V}$ is the set of all acyclic maps, then we say that a convexity associated with $\mathcal{V}$ is *acyclic*.

COROLLARY 2.41. *Let $Y \in ANES(compact)$ be Hausdorff acyclic compact. Then there exists a full acyclic convexity $\kappa = (Y, \{C_V\}, Y)$ associated with $\mathcal{A} = \{\{y\} : y \in Y\}$, $conv(\{y\}) = \{y\}$ for all $y \in Y$.*

Applying a similar construction to $Y \in AES(compact)$, we obtain the following.

LEMMA 2.42. *Let $Y \in AES(compact)$ be compact. Then for any $V \in \mathcal{B}$, there exists a (single-valued) uniformly continuous function $C_V : \Delta(Y) \to Y$ such that C_V and C are V-near on Q.*

THEOREM 2.43. *Let $Y \in AES(compact)$ be Hausdorff compact. Then there exists a (strong) full simplicial convexity $(Y, \{C_V\}, Y)$.*

Clearly this construction provides a continuous convexity (Y, C, Y) for any ANR. The obvious examples of $Y \in ANES(compact) \backslash ANR$ are the closure of the graph of $y = \sin \frac{1}{x}$, $0 < x \leq 1$, and the “comb space”. Notice that these spaces are not $ANES(compact)$ in the sense of Gorniewicz and Granas [**27**].

3. Selection and Fixed-Point Theorems.

3.1. Introduction. We shall use the fact that presence of convex combinations reduces the question of existence of fixed points for a certain class of multifunctions on topological spaces to the question of existence of fixed points of multifunctions on simplexes (see [**4, 57**]). This enables us to use the Brouwer fixed point theorem for Δ_n or the theorem below that is contained in Corollary 2.3 of Gorniewicz [**26**]. He defines a class of multifunctions called admissible, proves that a composition of any number of admissible multifunctions between arbitrary spaces is admissible and that an acyclic multifunction is admissible. Many fixed point theorems in the existing literature will be shown to be reducible to this theorem.

THEOREM 3.1. *Any admissible map (in the sense of Gorniewicz* [**24**]*) and, in particular, any composition of acyclic multifunctions, $F : \Delta_n \to \Delta_n$ has a fixed point.*

Another purpose of ours is to obtain fixed point theorems for topological spaces without linear or convex structure. To achieve this goal we need to show that a given topological space can be equipped with a convexity structure; that is, to prove existence of convex combination functions. This can be accomplished for a wide class of extension spaces without metrizability, and as a corollary we obtain a generalization of the following theorem due to Eilenberg and Montgomery [**21**] (as well as some recent results):

THEOREM 3.2 (Eilenberg-Montgomery Fixed-Point). *Let X be an acyclic compact ANR, and let $F : X \to X$ be an acyclic multifunction. Then F has a fixed point.*

The chapter is organized as follows. In Section 22, we present some definitions and supplementary results from general topology. In Section 23, we prove our continuous selection theorem (the proof is based on the standard partition of unity argument) and derive from it the Michael and Browder Selection Theorems. In Section 25, we consider classes of multifunctions with fixed-point conditions. In Section 26, we establish our fixed-point theorem and in Sections 27 and 28 show how to obtain from it the Kakutani and Browder Fixed-Point Theorems.

3.2. Preliminaries From General Topology. Recall that we fixed the uniform space Y with the minimal open uniform base $\mathcal{B}$ and selected the index set I with $|I| = \omega$ such that ω is larger than $2^{|X|}$ for each space X involved.

Presenting necessary definitions we mostly follow Engelking [**22**].

DEFINITION 3.3. The *weight* $w(Y)$ is the cardinality of $\mathcal{B}$. Let $\varphi(Y)$ be the largest cardinal number $\mu \leq \omega$ such that the intersection of any family of elements of $\mathcal{B}$ whose cardinality is less then μ contains an element of $\mathcal{B}$.

Note that if $\mathcal{B} = \{U_\beta : \beta < \mu\}$, where μ is an infinite ordinal, is ordered by inclusion (in this case Y is called a *μ-metrizable* space [**33**]), then $\varphi(Y) = w(Y)^+$, where α^+ stands for the least cardinal number larger that α.

DEFINITION 3.4. For a topological space X, the *Lindelöf number* $l(X)$ is the least cardinal number λ such that every open cover of X has a subcover whose cardinality is less than λ ("at most" in [**22**]). Let $l'(X)$ be the largest cardinal number $\mu \leq \omega$ such that any open cover of X whose cardinality is less than μ has a finite subcover. Let $p(X)$ be the largest cardinal number $\kappa \leq \omega$ such that any open cover of X whose cardinality is less than κ has a locally finite open refinement.

Then X is known [**64**] as *finally λ-compact* (or *λ-Lindelöf* [**33**]) and *initially μ-compact*, respectively. In particular, when $l(X) \leq \aleph_1$, then X is a *Lindelöf space* (any open cover has a countable subcover), and when $l'(X) > \aleph_0$, the space X is called a *countably compact space* (any countable cover has a finite subcover). And for Hausdorff spaces, when $p(X) = \omega$ then X is *paracompact,* and when $p(X) > \aleph_0$, X is called *countably paracompact* [**22**, 5.5.2, p. 316] (any countable cover has a locally finite refinement).

Definition 3.5. For the uniform space Y, let $l_u(Y)$ be the least cardinal number λ such that for every $V \in \mathcal{B}$, the cover $\{B(x,V) : x \in Y\}$ has a subcover whose cardinality is less than λ.

If $l_u(Y) = \aleph_0$ then Y is known as *totally bounded* or *precompact* [**40**].

The properties below follow directly from the above definitions.

Proposition 3.6. *(a) If X is compact, then $l(X) = \aleph_0$ and $l'(X) = \omega$.*
(b) X is compact if and only if $l'(X) \geq l(X)$.
(c) If Y is discrete, then $l(Y) = |Y|^+$ and $\varphi(Y) = \omega$.
(d) $l_u(Y) \leq l(Y)$.
(e) $p(X) \geq l'(X)$.
(f) $p(X) \geq \aleph_0$.
(h) $\varphi(Y) \geq \aleph_0$.

The proof of the following is familiar.

Lemma 3.7. *Let μ be a cardinal number and suppose $l(Y) \leq \mu$. Suppose also that a family $\{F_i : i \in J\}$ of closed sets has the μ-intersection property, i.e., if $K \subset J$ and $\mid K \mid < \mu$, then $\cap_{i\in K} F_i \neq \emptyset$. Then the family $\{F_i : i \in J\}$ has a nonempty intersection.*

Proof. If $\bigcap_{i\in J} F_i = \emptyset$ then $\gamma = \{U_i = Y \setminus F_i : i \in J\}$ is an open cover of Y. Therefore, by definition of $l(Y)$, γ has a subcover $\{U_i : i \in K\}$, $K \subset J$, with $\mid K \mid < l(Y)$. Hence we have $\bigcap_{i\in K} F_i = \emptyset$ and $\mid K \mid < \mu$. This contradicts the μ-intersection property. □

Theorem 3.8. *Let $\mu = l(Y)$ and let J be a directed set satisfying the following:*
if $K \subset J$ and $\mid K \mid < \mu$ then K has an upper bound.
Then any net $\{y_i : i \in J\}$ in Y has a convergent subnet.

Proof. For each $i \in J$, let $F_i = \overline{\{y_j : j \geq i\}}$. Then $\{F_i : i \in J\}$ consists of nonempty closed sets and is decreasing, i.e., $i < j \Rightarrow F_i \supset F_j$. Take $K \subset J$ with $\mid K \mid < \mu$. Then K has an upper bound, so $\bigcap_{i\in K} F_i \neq \emptyset$. Therefore $\{F_i : i \in J\}$ has the μ-intersection property. Then, by the preceding lemma, there is some $y \in \bigcap_{i\in J} F_i$, which is a cluster point of the net. Then, by [**22**, Theorem 1.6.1], some subnet converges to y. □

Corollary 3.9. *Suppose $\mathcal{B} = \{U_i : i \in J\}$ is partially ordered by inclusion and $\varphi(Y) \geq l(Y)$. Then any net $\{y_i : i \in J\}$ in Y has a convergent subnet.*

Proof. Let $\mu = l(Y)$. Consider $K \subset I$ with $\mid K \mid < \mu$. Then we have $\mid K \mid < \varphi(Y)$. From the definition of $\varphi(Y)$ it follows that $\bigcap_{i\in K} U_i$ contains an element of $\mathcal{B}$, so K has an upper bound in J. The conclusion now follows from Theorem 22.6. □

3.3. The Main Selection Theorems. Since Theorems 11.1 - 11.4 deal with two types of multifunctions (u.s.c. and l.s.c.), we shall consider maps $T : X \to Y$ and $R : Z \to X$ of either kind between the two spaces ($Z = Y$) and a fixed point of their composition $T \circ R : Y \to Y$. The diagram below illustrates proofs in Sections 23 and 26.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow^{f} & \nwarrow^{R} & \| \\ \Delta_n & \xrightarrow{C_V(\cdot,a)} & Z, \end{array} \tag{3.1}$$

If a multifunction $T : X \to Y$ has admissible images then its *convex hull* $conv(T) : X \to Y$ is given by

$$conv(T)(x) = conv(T(x)), \quad x \in X.$$

Michael proves his selection theorem for his convex structures [**51**] by considering a sequence of "almost continuous" selections, while for locally convex topological vector spaces he constructs [**50**] a sequence of continuous "almost selections". The former yields sharper selection results, the latter requires an additional restriction on the convexity (some neighborhood of an admissible set is admissible, as in Proposition 19.2), but allows us to proceed directly to fixed point theorems.

THEOREM 3.10 (Almost Selection). *Let X be a normal topological space, $\kappa = (Y, \{C_V\}, Z)$ a convexity, Y' a subset of Y. Suppose also that:*

(i) $T : X \to Y$ is admissible-valued l.s.c. and $T(x) \cap Y' \neq \emptyset$ for any $x \in X$,

(ii) $p(X) \geq l_u(Y')$.

Then for any $U \in \mathcal{B}$, there exist $V \in \mathcal{B}$, $a \in (Y')^\omega$, and a continuous function $f : X \to \Delta_\omega$ satisfying the following conditions:

(a) for any $x \in X$, there is an open neighborhood G of x such that $f(G) \subset \Delta_n \subset \Delta_\omega$ for some $n \geq 0$,

(b) $f(X) \times \{a\} \subset Q'$, and

(c) $C_V(f(x), a) \subset B(conv(T)(x), U)$ for all $x \in X$.

If, in addition, we have $C_V = C$ for each $V \in \mathcal{B}$, and if Y' is admissible, then

$$C(f(x), a) \subset \overline{conv}(Y').$$

If, moreover,

(ii′) $l'(X) \geq l_u(Y')$,

then we have

(a′) $f(X) \subset \Delta_n \subset \Delta_\omega$ for some $n \geq 0$.

Proof. Let $U \in \mathcal{B}$. Then condition (E) reads as follows: there exist $V \in \mathcal{B}$ and $W \in \mathcal{B}$ such that

(3.2) $$C_V(\Delta(B(A, W))) \subset B(conv(A), U) \text{ for all admissible } A \subset Y.$$

Let $M = \{B(y, W) : y \in Y'\}$. This is an open cover of Y', so by definition of $l_u(Y')$, M has a subcover M' with $|M'| < l_u(Y')$. But $T : X \to Y$ is l.s.c., so $N = \{T^{-1}(G) : G \in M'\}$ consists of open sets. Moreover, if $x \in X$ then $T(x) \cap G \neq \emptyset$ for some $G \in M'$, because M' is a cover of Y' and $T(x) \cap Y' \neq \emptyset$. Hence x belongs to $T^{-1}(G)$, so N is an open cover of X. By (ii), we have $|N| = |M'| < l_u(Y') \leq p(X)$. Therefore, by definition of $p(X)$, N has a locally finite open refinement N'. Since $|I| = \omega > 2^{|X|}$, we can assume that $N' = \{Q_k : k \in I\}$ ($Q_k = \emptyset$ for some $k \in I$). From the definitions of $M,\ N,\ N'$, it follows that for all $k \in I$,

$$Q_k \subset T^{-1}(G_k), \text{ where } G_k = B(a_k, W) \text{ for some } a_k \in Y'$$

(here we assign an index to G_k and a_k according to this inclusion). Then let

$$a = (a_i)_{i \in I} \in (Y')^\omega.$$

From the fact that X is normal, it follows from Michael's Lemma 12.1 that there exists a partition of unity subordinate to N', i.e., there are continuous functions $f_k : X \to [0, 1],\ k \in I$, satisfying

$$f_k(x) = 0 \text{ for any } x \notin Q_k,\ k \in I, \text{ and}$$
$$\textstyle\sum_{k \in I} f_k(x) = 1 \text{ for any } x \in X.$$

Now let

$$f(x)=\sum_{k\in I} f_k(x)e_k,$$

where $e_k,\ k\in I$, are the vertices of Δ_ω. Then $f: X\to\Delta_\omega$ is a continuous function. Since N' is locally finite, for each $x\in X$, there are a neighborhood G of x and a finite set $S\subset I$ such that $f_k|_G$ is nonzero only for $k\in S$. Therefore we have $f(G)\subset\Delta_S$, so (a) is satisfied. Moreover, if (ii′) holds, then N' is finite and so (a′) holds.

Take $x\in X$. Let

$$K=\{k\in I: T(x)\cap G_k\neq\emptyset\},$$
$$A_K=\{a_k: k\in K\}\subset Y'.$$

If $k\in I$ is such that $f_k(x)\neq 0$ then $x\in Q_k\subset T^{-1}(G_k)$, or $T(x)\cap G_k\neq\emptyset$. Hence $k\in K$. By definition of f, this implies that

$$f(x)=\sum_{k\in I} f_k(x)e_k=\sum_{k\in K} f_k(x)e_k\in\Delta_K, \tag{3.3}$$

where $e_k, k\in I$, are the unit vectors of $[0,1]^\omega$. Next, consider

$$\begin{aligned}\Delta_K\times\{a\} &=\{(d,a): i\notin K\Rightarrow d_i=0\}\\ &=\{(d,a): d_i\neq 0\Rightarrow a_i\in A_K\}\\ &\subset\Delta(A_K).\end{aligned} \tag{3.4}$$

Recall that $G_k=B(a_k,W),\ k\in I$, so by definition of $K,$ we have $a_k\in B(T(x),W)$ for all $k\in K$, or

$$A_K\subset B(T(x),W)\cap Y'. \tag{3.5}$$

From (23.3)-(23.5), it follows that

$$f(x)\times\{a\}\in\Delta_K\times\{a\}\subset\Delta(A_K)\subset\Delta(B(T(x),W)\cap Y'). \tag{3.6}$$

Since $T(x)$ is admissible, $\Delta(B(T(x),W))\subset Q'$ (condition (c8) of Section 13). Hence by (23.6), we have $f(X)\times\{a\}\subset Q'$ and, therefore, $C_V(f(x),a)$ is well defined. Further, consider

$$\begin{aligned}C_V(f(x),a) &\subset C_V(\Delta(B(T(x),W))) &&\text{by (23.6)}\\ &\subset B(conv(T)(x),U) &&\text{by (23.2).}\end{aligned}$$

To finish the proof, we notice that by (23.6), $f(x)\times\{a\}\in\Delta(Y')$. Therefore if for each $V\in\mathcal{B}$, $C_V=C$, then by Proposition 13.2, we have

$$C(f(x),a)\subset C(\Delta(Y'))\subset\overline{conv}(Y').\ \square$$

Corollary 3.11 (Continuous Almost Selection). *Let X be a normal topological space, $\kappa=(Y,\{C_V\},Z)$ a continuous convexity, Y' a subset of Y. Suppose also that*

(i) $T: X\to Y$ is l.s.c. with admissible images and $T(x)\cap Y'\neq\emptyset$ for any $x\in X$,

(ii) $p(X)\geq l_u(Y')$.

Then for any $V\in\mathcal{B}$, there exists a continuous V-almost selection for the multifunction $conv(T): X\to Z$, i.e., there is a continuous function $g: X\to Z$ such that

$$g(x)\in B(conv(T(x)),V)\ \text{for all}\ x\in X.$$

If, moreover, we assume that for each $V \in \mathcal{B}$, $C_V = C$, and Y' is admissible, then we have

$$g(X) \subset \overline{conv}(Y').$$

Proof. All the conditions of Theorem 23.1 are satisfied, so conclusions (a), (b), (c) hold. Then let

$$g(x) = C_V(f(x), a) \text{ for } x \in X.$$

The function g is well defined by (b). According to (a), for each $x \in X$, there is a neighborhood G of x such that $f(G) \subset \Delta_n$ for some n-simplex $\Delta_n \subset \Delta_\omega$. Since the convexity is continuous, $\mathcal{V}$ is the class of all continuous maps. So condition (D) turns into the following: for any $V \in \mathcal{B}$, $a \in Y^\omega$, if $\Delta_n \times \{a\} \subset Q'$, then the multifunction $C_V(\cdot, a) : \Delta_n \to Z$ is continuous. Therefore g is continuous at x. □

Definition 3.12. We say that the convexity $\kappa = (Y, \{C_V\}, Z)$ has a *convex uniform base* $\mathcal{B}$ if

$$y \in Y,\ U \in \mathcal{B},\ D \in \mathcal{C} \Rightarrow B(y, U) \cap D \in \mathcal{C}$$

Theorem 3.13 (Continuous Selection). *Let X be a normal topological space, Y be a complete uniform space, $\kappa = (Y, \{C_V\}, Z)$ a continuous convexity with a countable convex uniform base $\mathcal{B}$, and suppose that the uniform topology of Y is finer than the topology of Z. Suppose also that*

(i) $T : X \to Y$ is l.s.c. with nonempty convex images,

(ii) $p(X) \geq l_u(Y)$.

Then the multifunction $\overline{T} : X \to Z$ has a continuous selection, i.e., there is a continuous map $g : X \to Z$ such that

$$g(x) \in \overline{T(x)} \text{ for all } x \in X \text{ (closure in } Y).$$

Proof. Let $\mathcal{B} = \{U_1, U_2, \ldots\}$. Without loss of generality we can assume that

$$2U_{n+1} \subset U_n,\ n = 1, 2, \ldots. \tag{3.7}$$

Then according to Corollary 23.2 (with $Y' = Y$), for any nonempty convex-valued l.s.c. map $G : X \to Y$, for any $U \in \mathcal{B}$, there is a continuous $g : X \to Z$ with

$$g(x) \in B(G(x), U) \text{ for all } x \in X. \tag{3.8}$$

We inductively construct a sequence of continuous functions $g_n : X \to Z,\ n = 1, 2, \ldots$, such that

$$g_n(x) \in B(T(x), U_{n+1}) \text{ for all } x \in X,\ n = 1, 2, \ldots, \tag{3.9}$$

$$g_n(x) \in B(g_{n-1}(x), U_{n-1}) \text{ for all } x \in X,\ n = 2, 3, \ldots. \tag{3.10}$$

By (23.8), there is a g_1 so that (23.9) holds for $n = 1$. Assume that we have constructed $g_1, \ldots, g_{n-1}$ satisfying these conditions. Then let

$$G(x) = B(g_{n-1}(x), U_n) \cap T(x). \tag{3.11}$$

By Proposition 2.5 of [**49**], G is l.s.c.. By (23.9) for $n - 1$, we have $g_{n-1}(x) \in B(T(x), U_n)$, so $G(x)$ is nonempty, and it is convex because the base in convex. Therefore by (23.8), there is a continuous map $g_n : X \to Z$ with

$$g_n(x) \in B(G(x), U_{n+1}) \text{ for all } x \in X. \tag{3.12}$$

Then there is a $y \in G(x)$ such that

$$(g_n(x), y) \in U_{n+1}. \tag{3.13}$$

By (23.11), we have $G(x) \subset T(x)$, so from (23.12) we obtain the following:

$$g_n(x) \in B(T(x), U_{n+1}). \tag{3.14}$$

Therefore g satisfies (23.9). From (23.11), it also follows that $G(x) \subset B(g_{n-1}(x), U_n)$. Therefore we have

$$(y, g_{n-1}(x)) \in U_n. \tag{3.15}$$

Now, from (23.13), (23.15), and (23.7), it follows that

$$(g_n(x), g_{n-1}(x)) \in U_{n-1}.$$

Hence (23.10) holds. Thus, we have constructed a sequence $\{g_n : n = 1, 2, \ldots\}$ satisfying the required conditions.

Now (23.7) implies that this is a Cauchy sequence. Therefore $\{g_n : n = 1, 2, \ldots\}$ converges to a map $g : X \to Z$. And from (23.9), it follows that $g(x) \in \overline{T(x)}$ for all $x \in X$. To finish the proof we observe that $g : X \to Z$ is the uniform limit of $\{g_n\}$ with respect to the uniformity of Y. Therefore g is continuous, since Y is finer than Z. □

Of course, the conditions of the theorem imply that Y is pseudometrizable [**22**, IV.4.1].

3.4. More Selection Theorems. This theorem has the two corollaries below, which may be called Michael and Browder-type selection theorems respectively.

THEOREM 3.14. *Let X be a (Hausdorff) paracompact space, $\kappa = (Y, \{C_V\}, Y)$ a continuous convexity, Y complete with a countable convex uniform base, $T : X \to Y$ l.s.c. with nonempty convex images. Then $\overline{T}$ has a continuous selection.*

Proof. In Theorem 23.4, we let $Z = Y$ and notice that $p(X) = \omega \geq l_u(Y)$, so condition (ii) of the theorem holds. □

If we replace the convexity $\kappa = (Y, \{C_V\}, Y)$ in this theorem with particular convex structures considered in Chapter 2 we obtain versions of the results discussed below (additional condition are given in **bold**).

(1): The Michael Selection Theorem 11.3 for Banach spaces.

A Banach space has a simplicial convexity by Proposition 16.3, and it has a countable convex base — the balls of radii $1/n, n = 1, 2, 3, ...$ □

(2): Theorem 1.3 of Michael [**51**]. Let E be a complete metric space with a Michael convex structure (see Definition 19.1), and let $\mathcal{D}$ be the family of non-empty M-admissible subsets of E. Let X be paracompact, and $\varphi : X \to E$ l.s.c. with values in $\mathcal{D}$. Then there exists a continuous $f : X \to E$ such that

$$f(x) \in \overline{conv^*(\varphi(x))} \text{ for all } x \in X.$$

We obtain this theorem with two additional assumptions: **(1) for any** $x \in X$, **any** $W \in \mathcal{B}$, **the set** $B(\varphi(x), W)$ **is M-admissible, (2) for any** $y \in E$, **any** $W \in \mathcal{B}$, **the set** $B(y, W)$ **is M-convex**. There is a continuous convexity on $Y = E$ (Proposition 19.2) as E has an M-convex structure. From (1) it follows that $\varphi(x)$ is admissible and by (2) it has a convex uniform base. The rest follows from Theorem 24.1 as $T(x) = conv^*(\varphi(x))$ is convex. □

(3): Theorem 3.3 of Horvath [**36**]. Let X be a paracompact topological space, $(Y, \{\Gamma_A\})$ a Horvath l.c. complete metric space (see Definition 17.3) and $T : X \to Y$ a l.s.c. map such that $\forall x \in X,\ T(x)$ is a nonempty closed H-convex set. Then there is a continuous selection for T.

By Definition 17.3 and Proposition 17.4, we have a continuous convexity and a countable convex uniform base – the balls of radii $1/n, n = 1, 2, 3, ...$. Then the theorem implies existence of a selection. □

(4): Part (2) of Theorem 3.5 of Van de Vel [**67**, p. 440] or part (b) of Theorem 4.3 of [**68**]. Let Y be a topological Van de Vel convex structure satisfying the S_4-axiom with compact polytopes and with connected V-convex sets (for definitions, see Section 18), and let d be a compatible metric on Y. If X is paracompact space, then each l.s.c. multifunction $X \to Y$ with convex and d-complete values admits a continuous selection.

We prove this under the additional assumption that the whole space Y **is complete**. By Proposition 18.4, there is a full continuous convexity (Y, C, Y). As in (2), by condition (4) of Definition 18.1, $(E2)$ holds, so $\mathcal{B}' = \{conv(B(y, W)) : y \in \varphi(X), W \in \mathcal{B}\}$ is a countable convex uniform base ($\mathcal{B}$ is a countable base of the metric uniformity). Note that we obtain this theorem by using its own corollary (therefore, not independently) to show how our selection theorem is related to Van de Vel's, see Section 18. □

Here (3) and (4) may be looked at as selection theorems for a l.s.c. map with convex range.

An example of a pair X, Y that satisfies conditions (ii) of Theorem 23.4 but is not covered by the Michael Selection Theorem: X is normal but not necessarily paracompact and Y is precompact. Another example: X is countably paracompact, or countably compact, such as the space W of all countable ordinal numbers, and Y is a separable Banach space, such as l_2 or $C[0, 1]$. Nedev [**54**] proved a Michael-type selection theorem for $X = W$ and Y a reflexive Banach space.

Theorem 3.15. *Let X be a (Hausdorff) paracompact space, and suppose (Y, C, Z) is a discrete convexity, and $T : X \to Y$ has nonempty admissible images and open fibers. Then $conv(T) : X \to Z$ has a continuous selection.*

Proof. In Corollary 23.2 we let Y be discrete and $Y' = Y$. Then $T : X \to Y$ is l.s.c., so condition (i) of the theorem holds. We also notice that $p(X) = \omega \geq l_u(Y)$, so (ii) also holds. Finally, we notice that an almost selection with respect to the discrete topology is actually a selection. □

If we replace the discrete convexity (Y, C, Z) in this theorem with particular convex structures considered in Chapter 2 we obtain versions of the following results.

(1): The Browder Selection Theorem 11.4 for topological vector spaces.

It follows from Proposition 16.8 that any topological vector space has a discrete convexity. Now the existence of a selection follows from the theorem. □

(2): Theorem 3.2 of Horvath [**36**]. Let X be a paracompact topological space, $(Z, \{\Gamma_A\})$ a Horvath space (see Definition 17.1), $S, P : X \to Z$ two maps such that:
(i) $\forall x \in X,\ S(x) \subset P(x)$,

(ii) $\forall x \in X$, $\forall$ finite $A \subset S(x)$, $\Gamma_A \subset P(x)$,
(iii) $X = \bigcup\{intS^{-1}(y) : y \in Z\}$.
Then P has a continuous selection $g : X \to Z$. Furthermore, if X is compact there is a finite subset $D \subset Z$ such that $g(X) \subset \Gamma_D$.

Let Y be the discrete uniform space on the set Z (so all subsets of Y are admissible). We define T by its graph: $Graph(T) = \{intS^{-1}(y) \times \{y\} : y \in Y\} \subset X \times Y$. Then $T : X \to Z$ has open fibers and nonempty images according to (iii). By Proposition 17.2, there is a discrete convexity (Y, C, Z). Hence by the theorem $conv(T)$ has a continuous selection and then so does P, because by (ii)

$$conv(T(x)) = conv^*(T(x)) \subset P(x) \text{ for all } x \in X. \ \square$$

(3): Van de Vel's version of the Browder Selection Theorem [**67**, IV.3.23, p. 450]. Let Z be a metrizable Van de Vel convex structure satisfying the S_4-axiom with connected V-convex sets and with compact polytopes (for definitions, see Section 18). Let X be a paracompact space and let $T : X \to Z$ be a multifunction such that the set $T^{-1}(x) \subset X$ is open for each $y \in Z$. Then the multifunction $x \longmapsto conv(T(x))$ has a continuous selection.

By Proposition 18.4, there is a full continuous convexity (Z, C, Z). Let Y be the discrete uniform space on the set Z with $\mathcal{A} = 2^Y$. Then (Y, C, Z) is a discrete continuous convexity. Hence T has a selection. $\square$

THEOREM 3.16. *Suppose $\kappa = (Y, \{C_V\}, Y)$ is a simplicial convexity and Y is complete metrizable space with a countable convex uniform base. Then any convex closed set $K \subset Y$ is an AR.*

Proof. Suppose X is a Hausdorff compact space, $B \subset X$ is closed, $f : B \to K$ is continuous. Define a multifunction $T : X \to K$ by

$$T(x) = \begin{cases} \{f(x)\} & \text{if } x \in B \\ K & \text{otherwise.} \end{cases}$$

Then T is l.s.c. with nonempty closed convex images. By Theorem 24.1, T has a selection $f' : X \to Y$, which in fact extends f. $\square$

3.5. Classes of Maps with Fixed Point Conditions. For any class $\mathcal{U}$ of multifunctions, let

$$\mathcal{U}_c = \{F = F_n \circ F_{n-1} \circ \ldots \circ F_1 : F_i \in \mathcal{U}\}, \text{and}$$
$$\mathcal{U}_c(X, Y) = \{F : X \to Y\} \cap \mathcal{U}_c.$$

Motivated by Ben-El-Mechaiekh and Deguire [**4**], Park and Kim [**57**] define an abstract class $\mathcal{U}$ of maps that helps reduce the fixed point problem to the one for multifunctions on simplexes. They assume that the following properties are satisfied:

(i) $\mathcal{U}$ contains all single-valued continuous functions,
(ii) each $F \in \mathcal{U}_c$ is u.s.c. with compact images,
(iii) each $F \in \mathcal{U}_c(\Delta_n, \Delta_n)$ has the fixed point property.

Examples of $\mathcal{U}$ are, for instance, the classes of all u.s.c. multifunctions with compact convex values in locally convex topological vector spaces, acyclic maps, and approachable maps, see [**4, 57**]. Park also defines another class of multifunctions: $F \in \mathcal{U}_c^\kappa(Y, X)$ if for any compact $K \subset Y$, there is a $G \in \mathcal{U}_c(K, X)$ such that $G(y) \subset F(y)$ for each $y \in K$.

Taking this one step further, we introduce the following.

Definition 3.17. Let X be a topological space, $\kappa = (Y, \{C_V\}, Z)$ be a convexity and Y' a subset of Y (or Z). Then the class $\mathcal{F}_\kappa(Y', X)$ is defined as the class of all multifunctions $F : Z \supset Y' \to X$ such that for any simplex $\Delta_n \subset \Delta_\omega$, any $a \in (Y')^\omega$, any $V \in \mathcal{B}$, and any continuous function $f : X \to \Delta_n$, the multifunction

$$f \circ F \circ C_V \mid_{\Delta_n \times \{a\}} : \Delta_n \to \Delta_n$$

has a fixed point.

Notice that this definition does not depend on the topology of Y and Z.

By Definitions 13.1 and 16.1, $C_V(\cdot, a) : \Delta_n \to Z$, $V \in B$, is continuous when $\kappa = (Y, \{C_V\}, Z)$ is a continuous convexity. Therefore the Brouwer Fixed Point Theorem implies (see diagram (23.1)):

Proposition 3.18. *If $\kappa = (Y, \{C_V\}, Z)$ is a continuous full convexity then $\mathcal{F}_\kappa(Y, X)$ contains all continuous functions $\psi : Z \to X$.*

Recall that if $\mathcal{V}$ is the class of all acyclic maps, then a convexity associated with $\mathcal{V}$ is called acyclic. Since all single-valued functions are admissible in the sense of Gorniewicz [**26**, Theorem III.2.7] (see Section 12) and the composition of admissible in the sense of Gorniewicz multifunctions is admissible in the sense of Gorniewicz, Theorem 21.1 implies

Proposition 3.19. *If $\kappa = (Y, \{C_V\}, Z)$ is a full acyclic convexity then $\mathcal{F}_\kappa(Y, X)$ contains all admissible in the sense of Gorniewicz (and, therefore, all acyclic) multifunctions $F : Z \to X$.*

Proposition 3.20. *If $\kappa = (Y, C, Y)$ is a full acyclic convexity and Y' is a closed convex subset of Y, then $\mathcal{F}_\kappa(Y', X)$ contains all u.s.c. multifunctions $F : Y' \to X$ such that $F(x)$ is compact and acyclic for all $x \in Y'$.*

Proof. The composition $f \circ F \circ C \mid_{\Delta_n \times \{a\}} : \Delta_n \to \Delta_n$ is admissible in the sense of Gorniewicz because $C(\Delta(Y')) \subset Y'$ by Proposition 13.2. So it is a composition of acyclic multifunctions and, by Theorem 21.1, has a fixed point. □

Proposition 3.21. *If $\kappa = (Y, \{C_V\}, Y)$ is a full continuous convexity then we have*

$$\mathcal{U}_c^\kappa(Y, X) \subset \mathcal{F}_\kappa(Y, X).$$

Proof. Let $F \in \mathcal{U}_c^\kappa(Y, X)$. For any $a \in Y^\omega$, any $V \in \mathcal{B}$, the map $C_V(\cdot, a) : \Delta_n \to Z$ is continuous. Hence $K = C_V(\Delta_n, a)$ is compact, so there is a $G \in \mathcal{U}_c(K, X)$ such that $G(y) \subset F(y)$ for all $y \in K$. By (i), $C_V(\cdot, a)$ and f are in $\mathcal{U}_c(K, X)$, for any continuous $f : X \to \Delta_n$. Hence we have $f \circ G \circ C_V(\cdot, a) \in \mathcal{U}_c(\Delta_n, \Delta_n)$. Then this map has the fixed point property by (iii), and so does $f \circ F \circ C_V(\cdot, a)$. Hence F belongs to $\mathcal{F}_\kappa(Y, X)$. □.

3.6. The Main Fixed Point Theorems.

Theorem 3.22 (Almost Fixed Point I). *Let X be a normal topological space, $\kappa = (Y, \{C_V\}, Z)$ a full convexity. Suppose also that the following holds:*

(i) $R \in \mathcal{F}_\kappa(Y, X)$,

(ii) $T : X \to Y$ is admissible-valued l.s.c.,

(iii) $l'(X) \geq l_u(Y)$.

Then the multifunction $conv(T) \circ R : Y \to Y$ has a U-almost fixed point for any $U \in \mathcal{B}$, i.e., there is a

$$y \in B(conv(T) \circ R(y), U).$$

Proof. By Theorem 23.1 with (ii′), for any $U \in \mathcal{B}$ there exist $V \in \mathcal{B},\ a \in Y^\omega$, and a continuous function $f : X \to \Delta_J$ with J finite such that

$$C_V(f(x), a) \subset B(conv(T)(x), U) \text{ for all } x \in X.$$

Since the convexity is full, we can define a map $\Psi : \Delta_J \to Y$ by

$$\Psi = C_V|_{\Delta_J \times \{a\}}.$$

Since $R \in \mathcal{F}_\kappa(Y, X)$, the multifunction $F = f \circ R \circ \Psi : \Delta_J \to \Delta_J$ has a fixed point, that is, there exists a $d \in f(R(\Psi(d)))$. Therefore, there exist such $x \in X$ and $y \in Y$ that

$$y \in \Psi(f(x)) \text{ and } x \in R(y).$$

Hence

$$y \in C_V(f(x), a) \subset B(conv(T)(x), U) \subset B(conv(T) \circ R(y), U). \ \square$$

The following is a version of the above theorem for non-self maps.

THEOREM 3.23 (Almost Fixed Point II). *Let X be a normal topological space, (Y, C, Z) a full convexity, Y' a closed convex subset of Y. Suppose also that the following holds:*

(i) $R \in \mathcal{F}_\kappa(Y', X)$,
(ii) $T : X \to Y$ is admissible-valued l.s.c. and $T(x) \cap Y' \neq \emptyset$ for any $x \in X$,
(iii) $l'(X) \geq l_u(Y')$.

Then the multifunction $conv(T) \circ R : Y' \to Y$ has a U-almost fixed point in Y' for any $U \in \mathcal{B}$, i.e., there is a

$$y \in B(conv(T) \circ R(y), U) \cap Y'.$$

Proof. We follow the proof above. By Theorem 23.1, for any $U \in \mathcal{B}$ there exist $V \in \mathcal{B},\ a \in (Y')^\omega$, and a continuous function $f : X \to \Delta_J$ with finite J such that for all $x \in X$

$$C(f(x), a) \subset B(conv(T)(x), U) \text{ and}$$
$$C(f(x), a) \subset \overline{conv}(Y') = Y'.$$

As above, we define a map $\Psi : \Delta_J \to Y'$ by

$$\Psi = C|_{\Delta_J \times \{a\}}.$$

(Ψ maps Δ_J into Y' by Proposition 13.2). Since $R \in \mathcal{F}_\kappa(Y', X)$, the multifunction $F = f \circ R \circ \Psi : \Delta_J \to \Delta_J$ is well defined and has a fixed point, that is, there exists a $d \in f(R(\Psi(d)))$. Therefore, there exist such $x \in X$ and $y \in Y'$ that

$$y \in \Psi(f(x)) \text{ and } x \in R(y),$$

so as above, we have $y \in B(conv(T) \circ R(y), U)$. $\square$

THEOREM 3.24 (Fixed Point I). *Let X be a normal topological space, $\kappa = (Y, \{C_V\}, Z)$ a full convexity. Suppose that the following conditions are satisfied:*

(i) $R \in \mathcal{F}_\kappa(Y, X)$,
(ii) $T : X \to Y$ is admissible-valued l.s.c.,
(iii) $S : Y \to Y$ is closed-valued u.s.c. and $conv(T) \circ R \subset S$,
(iv) $\varphi(Y) \geq l(Y),\ l'(X) \geq l_u(Y)$.

Then S has a fixed point.

Proof. Recall that $\mathcal{B} = \{U_i : i \in J\}$ is a uniform base for Y partially ordered by inclusion, i.e., $k > i \Rightarrow U_k \subset U_i$. Then J is a directed set. By Theorem 26.1, for any $i \in J$, there is a

$$y_i \in B(conv(T) \circ R(y_i), U_i).$$

Then (iii) implies that

$$y_i \in B(S(y_i), U_i). \tag{3.16}$$

Now we need to show that $y^* \in S(y^*)$ for some $y^* \in Y$. By Corollary 22.7, the net $\{y_i : i \in J\}$ has a convergent subnet. Therefore, we can simply assume that $y_i \to y^* \in Y$. Then for any $i \in J$, there exists a $k(i) \in J$ such that $y_k \in B(y^*, U_i)$ for all $k > k(i)$, or

$$y^* \in B(y_k, U_i) \text{ for all } k > k(i). \tag{3.17}$$

By (26.1), $y_k \in B(S(y_k), U_k)$ for all $k \in J$. Therefore

$$y_k \in B(S(y_k), U_i) \text{ for all } k > i. \tag{3.18}$$

Since $y_i \to y^*$ and the multifunction S is u.s.c., then for any $i \in J$, there exists a $j(i) \in J$ such that

$$S(y_k) \subset B(S(y^*), U_i) \text{ for all } k > j(i). \tag{3.19}$$

Now using (26.2), (26.3), (26.4), we obtain

$$y^* \in B(S(y^*), 3U_i) \text{ for all } i \in J.$$

Hence $y^* \in \overline{S(y^*)} = S(y^*)$, and the proof is complete. □

Similarly, we obtain the following fixed-point theorem for non-self maps.

THEOREM 3.25 (Fixed Point II). *Let X be a normal topological space, (Y, C, Z) a full convexity, Y' a closed convex subset of Y. Suppose that the following conditions are satisfied:*

(i) $R \in \mathcal{F}_\kappa(Y', X)$,
(ii) $T : X \to Y$ is admissible-valued l.s.c. and $T(x) \cap Y' \neq \emptyset$ for any $x \in X$,
(iii) $S : Y \to Y$ is closed-valued u.s.c. and $conv(T) \circ R \subset S$,
(iv) $\varphi(Y) \geq l(Y)$, $l'(X) \geq l_u(Y')$.

Then S has a fixed point in Y'.

Proof. We follow the proof of Theorem 26.3 almost without change. By Theorem 26.2, for any $i \in J$, there is a

$$y_i \in B(conv(T) \circ R(y_i), U_i) \cap Y'.$$

Then (iii) implies that

$$y_i \in B(S(y_i), U_i) \cap Y'.$$

Then since $\varphi(Y) \geq l(Y)$, by Corollary 22.7, the net $\{y_i : i \in J\}$ has a convergent subnet. And we can assume that $y_i \to y^* \in Y'$ because Y' is closed. The rest is identical to the argument in the proof of Theorem 26.3 via (26.2)-(26.4). □

Condition (iv) "distributes" compactness between X and Y, and it is clearly satisfied when

(a) Y is compact, or
(b) X is compact and Y is discrete.

In the next two sections we will see that case (a) with a simplicial convexity gives the Kakutani Theorem 11.1 and case (b) with a discrete convexity gives the Browder Theorem 11.2. But there are many cases "between" these two: for any infinite cardinal μ, let X be initially μ-compact and Y finally μ-compact and μ-metrizable.

3.7. Kakutani and Eilenberg-Montgomery Type Fixed Point Theorems. One can obtain Kakutani and Browder type theorems for spaces with generalized convexity by repeating arguments that work for topological vector spaces, see [**36, 37, 56, 58, 5, 66**]. But then the results apply only to convex-valued multifunctions. Therefore they are never stronger than the Eilenberg-Montgomery Theorem 21.2, which deals with acyclic-valued multifunctions. So we apply a version of the Eilenberg-Montgomery Theorem, Theorem 21.1, or use $\mathcal{F}_\kappa(X,X)$ to obtain sharper results.

THEOREM 3.26. *Let X be a (Hausdorff) compact topological space, $\kappa=(Y,\{C_V\},Y)$ a full simplicial convexity, $f:X\to Y$ a (single-valued) continuous map and $R\in\mathcal{F}_\kappa(Y,X)$ u.s.c.. Then $\overline{f\circ R}$ has a fixed point.*

Proof. In Theorem 26.3 we let $T=f,\ S=\overline{f\circ R}$. Then (i) and (ii) hold because the convexity is simplicial: $conv(\{y\})=\{y\}$. Also, since Y is compact, we have $l'(X)\geq\aleph_0=l(Y)$ and $\varphi(Y)\geq\aleph_0=l(Y)$, so (iv) holds. □

Replacing in this theorem $\kappa=(Y,\{C_V\},Y)$ with particular convex structures considered in Chapter 2 and replacing $\mathcal{F}_\kappa(X,X)$ with appropriate classes of maps we obtain versions of the following results (additional conditions are given in **bold**, superfluous conditions are given in *italic*).

(1): The Kakutani Theorem 11.1 for locally convex topological vector spaces.

If Y is a locally convex topological vector space then by Proposition 16.3, there is a full simplicial convexity $\kappa=(Y,C,Y)$. Also note that $\mathcal{F}_\kappa(Y,Y)$ (here $X=Y,f=id_Y$) contains all u.s.c. multifunctions with closed convex values, as convex sets are acyclic in locally convex topological vector spaces. Then any u.s.c. convex closed-valued map has a fixed point by the theorem. □

(2): Corollary 3 of McLinden [**48**]. Let X be nonempty compact subset of a *complete* convex subset Y of a locally convex Hausdorff topological vector space. If R is an acyclic multifunction from Y into X, then there exists some $z\in Y$ such that $z\in R(z)$.

Let $f:X\to Y$ be the inclusion. First, by Proposition 16.3, there is a full simplicial convexity $\kappa=(Y,C,Y)$. Second, $\mathcal{F}_\kappa(Y,X)$ contains all acyclic multifunctions. So R has a fixed point by the theorem. □

(3): Corollary to Theorem 6 of Horvath [**37**]. Let $(Y,\{\Gamma_A\})$ be a compact metric l.c.-space (see Definition 17.1) such that for any nonempty finite subset $A\subset Y$, the set Γ_A is closed. Then a nonempty closed H-convex valued u.s.c. mapping $R:Y\to Y$ has a fixed point.

In the theorem, we let $X=Y,f=id_Y$. By Proposition 17.4, there is a full simplicial convexity $\kappa=(Y,C,Y)$. But by Theorem 24.3, all convex sets are AR's. Hence R is acyclic, so it belongs to $\mathcal{F}_\kappa(Y,X)$. □

(4): Theorem 6.15 of Van de Vel [**67**, p. 498]. Let Y be a compact Hausdorff Van de Vel convex structure satisfying the S_4-axiom with connected convex sets (for definitions, see Section 18). If Y is properly locally convex (i.e., each point has a neighborhood base of V-convex open sets), *closure stable* (i.e., the closure of each V-convex set is V-convex), then any u.s.c. V-convex closed valued multifunction $R : Y \to Y$ has a fixed point.

In the theorem, we let $X = Y, f = id_Y$. By Proposition 18.4, there is a full continuous convexity $\kappa = (Y, C, Y)$. It is simplicial because all points are V-convex. Any V-convex subset is an AR by Proposition 18.3, hence R is acyclic and belongs to $\mathcal{F}_\kappa(Y, Y)$. □

See also results considered in Hadžić [**30**] for convex valued maps on admissible in the sense of Klee subsets of topological vector spaces.

(5): Theorem 1 of Hadžić [**31**]. Let $(W, \{\Gamma_A\})$ be a Horvath space (see Definition 17.1) with uniformity $\mathcal{L}$ and let K be a nonempty, H-convex and precompact subset of W. Let $T : K \to W$ be a l.s.c. map such that $T(x) \cap K \neq \emptyset$ for every $x \in K$, and $T(x)$ is H-convex for every $x \in K$. If $K \cup T(K)$ is a of generalized Zima type (see Definition 17.3) then T has a U-almost fixed point, for every $U \in \mathcal{L}$.

In Theorem 26.2, we let $X = Y = K \cup T(K)$. Therefore Y is precompact and (iii) holds. Also $Y' = K,\ R = Id_X$, so (i) holds. Assume that in this H-space **all points are convex** and that Y **is normal**. Then by Proposition 17.4, there a full simplicial convexity (Y, C, Y). So by Theorem 26.2, T has a fixed point. □

The next two results are Eilenberg-Montgomery type theorems, i.e., fixed point theorems for multivalued maps on topological spaces without linear or convex structure.

THEOREM 3.27. *Let $X \in ANES(compact)$ be acyclic (Hausdorff) compact, and let $F : X \to X$ be an u.s.c. closed-valued admissible in the sense of Gorniewicz multifunction. Then F has a fixed point.*

Proof. By Corollary 20.5, there exists an acyclic convexity on X with $conv(\{x\}) = \{x\},\ x \in X$. Therefore, by Proposition 25.3, $F \in \mathcal{F}_\kappa(X, X)$. Then by Theorem 27.1, F has a fixed point. □

The Eilenberg-Montgomery Theorem 21.2 immediately follows because any ANR is an $ANES(compact)$ and any acyclic map is admissible in the sense of Gorniewicz (see Section 12). The same statement is true for non-ANR spaces like "the comb space", which shows that using approximative convexity multifunctions $\{C_V : V \in \mathcal{B}\}$ instead of C makes a difference.

This theorem is equivalent to the "acyclic" version of Lefschetz-type Theorem 3 of Gorniewicz and Granas [**27**], except for the difference in the definition of ANES mentioned in Section 20, so that theorem does not include "the comb space".

THEOREM 3.28. *Let $X \in AES(compact)$ be compact, and $F \in \mathcal{U}_c(X, X)$. Then F has a fixed point.*

Proof. By Theorem 20.7, there exists a continuous convexity on X with $conv(\{x\}) = \{x\},\ x \in X$. Therefore, according to 25.5, $F \in \mathcal{F}_\kappa(X, X)$. Then by Theorem 27.1, F has a fixed point. □

This theorem is exactly Theorem 6.6 of Ben-El-Mechaiekh and Deguire [**4**].

Using Brouwer's theorem instead of a homological result like the Eilenberg-Montgomery Theorem, we obtain the following.

Corollary 3.29. *Let $X \in AES(compact)$ be compact, and $g : X \to X$ be continuous. Then f has a fixed point.*

Proof. By Theorem 20.7, there exists a continuous convexity on X with $conv(\{x\}) = \{x\}$, $x \in X$. Therefore by Proposition 25.2, $g \in \mathcal{F}_\kappa(X, X)$. Then by Theorem 27.1, g has a fixed point. □

3.8. Browder-type Fixed Point Theorems.

Theorem 3.30. *Let X be compact, (Y, C, Z) be a discrete full convexity. Suppose that*
(i) $R \in \mathcal{F}_\kappa(Y, X)$,
(ii) $T : X \to Y$ is a multifunction with admissible images and open fibers.
Then the multifunction $conv(T) \circ R$ has a fixed point.

Proof. Notice that if Y is a discrete uniform space, then a multifunction $T : X \to Y$ has open fibers if and only if it is l.s.c.. Therefore conditions (i), (ii) of this theorem are exactly conditions (i), (ii) of Theorem 26.3. Condition (iii) follows from the fact that $S = conv(T) \circ R : Y \to Y$ is u.s.c. with respect to the discrete topology. Also, since X is compact and Y is discrete, we have $l'(X) = \omega \geq |Y| = l_u(Y)$ and $\varphi(Y) = \omega \geq |Y|^+ = l(Y)$, so (iv) holds. □

Replacing in this theorem (Y, C, Y) with particular convex structures considered in Chapter 2 and replacing $\mathcal{F}_\kappa(Y, X)$ with appropriate classes of maps we obtain the following results.

(1): The Browder Fixed Point Theorem 11.2 for topological vector spaces.

It follows from Proposition 16.8 that any topological vector space has a discrete convexity. Now the existence of a selection follows from the above theorem with $X = Y$ and $R = Id_X$ (T in the theorem is G in 11.2). □

(2): Theorem 7 of Browder [**7**]. Let Z be a convex subset of a topological vector space E, X a compact convex subset of a locally convex topological vector space F, and $R, P : Z \to X$ multivalued maps for which the following holds:
(1) R is a u.s.c. with nonempty closed convex values,
(2) for each $u \in Z$, $P(u)$ is an open subset of X and $P^{-1}(v)$ is a nonempty convex subset of Z for each $v \in X$.
Then there exists an element u_0 of Z such that $R(u_0) \cap P(u_0) \neq \emptyset$.

By Proposition 16.8, there is a discrete convexity $\kappa = (Y, C, Z)$. Then $T = P^{-1}$ is convex valued with respect to this convexity. But R is convex valued with respect to the convexity of F. Therefore R is acyclic because F is locally convex, so $R \in \mathcal{F}_\kappa(Y, X)$. Hence by the theorem, $conv(T) \circ R = T \circ R = P^{-1} \circ R$ has a fixed point u_0, so $R(u_0) \cap P(u_0) \neq \emptyset$. □

(3): Theorem 3 of McLinden [**48**]. Let T be a multifunction from a nonempty compact convex subset X of a Hausdorff topological vector space into a nonempty convex subset Z of a Hausdorff topological vector space. Assume that $T(x)$ is a nonempty convex set for each $x \in X$ and that $T^{-1}(z)$ is open for each $z \in Z$. If R is an acyclic multifunction from Z into X, then there exist some $\overline{x} \in X$ and $\overline{z} \in Z$ such that

$$\overline{x} \in R(\overline{z}) \text{ and } \overline{z} \in T(\overline{x}).$$

By Proposition 16.8, there is a discrete convexity $\kappa = (Y, C, Z)$. Now since all acyclic multifunctions belong to $\mathcal{F}_\kappa(Y, Y)$, we can apply the theorem, so $conv(T) \circ R = T \circ R$ has a fixed point $\overline{z}$. □

(4): Theorem 4.3 of Horvath [**36**]. Let X be a compact topological space, $(Y, \{\Gamma_A\})$ a Horvath space (see Definition 17.1), $S, P : X \to Y$ two maps such that:

(i) $\forall x \in X,\ S(x) \subset P(x)$,
(ii) $\forall x \in X,\ \forall$ finite $A \subset S(x)$, $\Gamma_A \subset P(x)$,
(iii) $X = \bigcup\{intS^{-1}(y) : y \in Y\}$.

Then for any continuous function $g : Y \to X$ there is a $y_0 \in Y$ such that $y_0 \in Pg(y_0)$.

By Proposition 17.2, there is a discrete convexity $\kappa = (Y, C, Z)$. In Theorem 28.1, we put $R = g \in \mathcal{F}_\kappa(Y, X)$, and define $T : X \to Y$ by its graph: $Graph(T) = \bigcup\{intS^{-1}(y) \times \{y\} : y \in Y\}$. Then T is defined on the whole X by (ii) and has open fibers. Hence $conv(T) \circ R$ has a fixed point. Therefore $P \circ g$ also has a fixed point, because by (ii) we have

$$conv(T) \circ R(x) = conv^*(T(g(x))) \subset P \circ g(x). \ \square$$

(5): Theorem 3.1 of Ding and Tarafdar [**15**]. Let X be nonempty compact and $(Z, \{\Gamma_A\})$ be a Horvath space, $R : Z \to X$ be u.s.c. compact acyclic valued and $P : X \to Z$ be a map satisfying

(a) for any $x \in X,\ P(x)$ is H-convex,
(b) for any $y \in Z$, there is an open $O_y \subset G^{-1}(y)$ such that $X \subset \bigcup_{y \in Z} O_y$.

Then $P \circ R$ has a fixed point.

The proof is similar to the one above: $Graph(T) = \bigcup\{O_y \times \{y\} : y \in Y\}$, but instead of a continuous function g we have an acyclic multifunction R. □

(6): Browder-type Fixed Point Theorem of Van de Vel [**67**, IV.6.28, p. 506]. Let Y be a metrizable Van de Vel convex structure satisfying the S_4-axiom with connected V-convex sets and with compact polytopes (for definitions, see Section 18). Let $T : Y \to Y$ be a multifunction such that the set $T^{-1}(x) \subset Y$ is open for each $y \in Y$. Then T has a fixed point.

By Proposition 18.4, there is a discrete convexity $\kappa = (Y, C, Z)$. Let $X = Y$ and $R = id_Y \in \mathcal{F}_\kappa(Y, X)$. Then the theorem implies the existence of a fixed point. □

Note that all the results mentioned in Sections 27 and 28 extend either Theorem 11.1 or 11.2 but not both, therefore Theorem 26.3 contains them properly.

As the Eilenberg-Montgomery Theorem 21.2 is a topological version of the Kakutani Theorem 11.1, the following may serve as a topological version of the Browder Theorem 11.2.

Corollary 3.31. *Let X be nonempty contractible compact and $F : X \to X$ be a multifunction with open fibers satisfying*

$$\text{for any open } U \subset X, \ \bigcap_{x \in U} F(x) \text{ is contractible or empty.}$$

Then F has a fixed point.

Proof. We construct a Horvath space as follows. For any finite $A \subset X$, let $P(A) = \bigcap_{y \in A} F^{-1}(y)$ (it is open) and let

$$\Gamma_A = \begin{cases} \bigcap_{x \in P(A)} F(x) & \text{if } P(A) \neq \emptyset, \\ X & \text{otherwise.} \end{cases}$$

Then F is H-convex valued and it has a fixed point by Theorem 28.1 with $R = Id$. □

References

[1] N. Aronszajn and P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric spaces, *Pacific J. Math.* **6** (1956), 405-439.

[2] C. Bardaro and R. Ceppitelli, Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities, *J. Math. Anal. Appl.* **132** (1988), 484-490.

[3] E. G. Begle, The Vietoris Mapping Theorem for bicompact spaces, *Ann. of Math.* **81** (1965), 82-99.

[4] H. Ben-El-Mechaiekh and P. Deguire, Approachability and fixed points for nonconvex set-valued maps, *J. Math. Anal. Appl.* **170** (1992), 477-500.

[5] R. Bielawski, Simplicial convexity and its applications, *J. Math. Anal. Appl.* **127** (1987), 155-171.

[6] G. E. Bredon, "Topology and Geometry", Springer-Verlag, 1993.

[7] F. E. Browder, The fixed point theory of multivalued mappings in topological vector spaces, *Math. Ann.* **177** (1968), 283-301.

[8] R. F. Brown, "The Lefschetz Fixed Point Theorem", Scott-Foresman, Chicago, 1971.

[9] R. F. Brown and H. Schirmer, Nielsen coincidence theory and coincidence-producing maps for manifolds with boundary, *Topology Appl.* **46** (1992), 65-79.

[10] R. F. Brown and H. Schirmer, Correction to "Nielsen coincidence theory and coincidence-producing maps for manifolds with boundary", *Topology Appl.* **67** (1995), 233-234.

[11] D. W. Curtis, Applications of a selection theorem to hyperspace contractibility, *Pacific J. Math.* **37** (1985) 4, 747-759.

[12] V. P. Davidyan, Coincidence points of two maps, *Russian Acad. Sci. Sb. Math.* **40** (1980), 205-210

[13] V. P. Davidyan, On coincidence of two maps for manifolds with boundary, *Russian Math. Surveys* **38** (1983) 2, 176.

[14] A. Dawidowicz, Spherical maps, *Fund. Math.* **127** (1987) 3, 187-196.

[15] X. P. Ding and E. Tarafdar, Some coincidence theorems and applications, *Bull. Austral. Math. Soc.* **50** (1994), 73-80.

[16] A. Dold, Fixed point index and fixed point theorem for Euclidean neighborhood retracts, *Topology* **4** (1965), 1-8.

[17] A. Dold, The fixed point transfer of fiber-preserving maps, *Math. Z.* **148** (1976), 215-244.

[18] A. Dold, A coincidence-fixed point index, *Einseign. Math.* (2) **24** (1978), 41-53.

[19] A. Dold, "Lectures on Algebraic Topology", Springer-Verlag, 1980.

[20] A. N. Dranishnikov, Absolute extensors in dimension n and dimension-raising n-soft maps, *Russian Math. Surveys* **39** (1984) 5, 63-111.

[21] S. Eilenberg and D. Montgomery, Fixed point theorems for multi-valued transformations, *Amer. J. Math.* **68** (1946), 214-222.

[22] R. Engelking, "General Topology," Second Edition, Heldermann Verlag, Berlin, 1989.

[23] I. L. Glicksberg, Further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points, *Proc. Amer. Math. Soc.* **3** (1952), 170-174.

[24] L. Gorniewicz, Homological methods in fixed point theory of multivalued maps, *Dissertationes Math. (Rozprawy Mat.)* **129** (1976), 1-71.

[25] L. Gorniewicz, Fixed point theorems for mutivalued maps of subsets of Euclidean spaces, *Bull. Acad. Polon. Sci.* **27** (1979), 111-115.

[26] L. Gorniewicz and A. Granas, Some general theorems in coincidence theory I, *J. Math. Pures Appl.* **60** (1981), 361-373.

[27] L. Gorniewicz and A. Granas, Topology of morphisms and fixed point problems for set-valued maps, *in* "Fixed Point Theory and Applications"(M.A. Thera and J.-B. Baillon, Eds.), pp. 173-191, Longman Scientific and Technical, Essex, 1991.
[28] V. G. Gutev, A fixed point theorem for UV^n uscos maps, *Proc. Amer. Math. Soc.* **124** (1996) 3, 945-952.
[29] J. Gwinner, On fixed points and variational inequalities - a circular tour, *Nonlin. Anal., Theory, Meth., Appl.* **5** (1981) 5, 565-583.
[30] O. Hadžić, "Fixed Point Theory in Topological Vector Spaces", Novi Sad, 1984.
[31] ________, Almost fixed point and best approximation theorems in H-spaces, *Bull. Austral. Math. Soc.* **53** (1996), 447-454.
[32] B. Halpern, A general coincidence theory, *Pac. J. Math.* **77** (1978) 2, 451-471.
[33] M. J. Harris, κ-Lindelöf κ-stratifiable spaces are κ-metrizable, *Topology Appl.* **73** (1996), 295-298.
[34] H. Hopf, Uber die algebraische Anzahl von Fixpunkten, *Math. Z.* **29** (1929), 493-524.
[35] C. Horvath, Some results on multivalued mappings and inequalities without convexity, *in* "Nonlinear and Convex Analysis" (B. L. Lin and S. Simons, Eds.), pp. 99-106, Marcel Dekker, 1987.
[36] ________, Contractibility and generalized convexity, *J. Math. Anal. Appl.* **156** (1991), 341-357.
[37] ________, Extension and selection theorems in topological spaces with generalized convexity structure, *Ann. Fac. Sci. Toulouse* **2** (1993) 2, 253-269.
[38] C. Horvath and J. V. L. Ciscar, Maximal elements and fixed points for binary relations on topological ordered spaces, *J. Math. Econ.* **25**(1996), 291-306.
[39] P. Husain and E. Tarafdar, A selection and a fixed-point theorem and an equilibrium point of an abstract economy, *Inter. J. Math. Math. Sci.* **16** (1995) 1, 179-184.
[40] I. M. James, "Introduction to Uniform Spaces," Cambridge University Press, Cambridge, 1990.
[41] D. S. Kahn, An example in Čech cohomology, *Proc. Amer. Math. Soc.* **16** (1969), 584.
[42] S. Kakutani, A generalization of Brouwer's fixed point theorem, *Duke Math.* J. **8** (1941), 457-459.
[43] K. Keimel and A. Wieczorek, Kakutani property of the polytopes implies Kakutani property of the whole space, *J. Math. Anal. Appl.* **130** (1988), 97-109.
[44] W. Kryszewski, Remarks on the Vietoris Theorem, *Topol. Methods Nonlin. Anal.* **8** (1996) 2, 383-405.
[45] Ky Fan, Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. USA* **38** (1952), 121-126.
[46] S. Lefschetz, Intersections and transformations of complexes and manifolds, *Trans. Amer. Math. Soc.* **28** (1926), 1-49.
[47] S. Lefschetz, "Algebraic Topology", American Mathematical Society Colloquium, Vol. 27 (Amer. Math. Soc., Providence, RI, 1942).
[48] L. McLinden, Acyclic multifunctions without metrizability, manuscript.
[49] E. Michael, Selected selection theorems, *Amer. Math. Monthly* **63** (1956), 233-238.
[50] ________, Continuous selections I, *Ann. Math.* **63** (1956), 361-382.
[51] ________, Convex structures and continuous selections, *Canadian J. Math.* **11** (1959), 556-575.
[52] K. Mukherjea, Coincidence theory for manifolds with boundary, *Topology Appl.* **46** (1992), 23-39.
[53] M. Nakaoka, Coincidence Lefschetz number for a pair of fiber preserving maps, *J. Math. Soc. Japan* **32** (1980), No. 4, 751-779.
[54] S. Nedev, A selection example, *C. R. Acad. Bulgare Sci.* **40**:11 (1987), 13-14.
[55] B. O'Neill, A fixed point theorem for multivalued functions, *Duke Math. J.* **24** (1957), 61-62.
[56] S. Park, Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps, *J. Korean Math. Soc.* **31** (1994), 493-519.
[57] S. Park and H. Kim, Coincidence theorems for admissible multifunctions on generalized convex spaces, *J. Math. Anal. Appl.* **197** (1996), 173-187.
[58] L. Pasicki, A fixed point theory and some applications of weeds, *Opuscula Math.* **7** (1990), 5-96.
[59] S. N. Patnaik, Fixed points of multivalued transformations, *Fund. Math.* **65** (1969) 345-349.
[60] D. Repovs and P. V. Semenov, Michael's theory of continuous selections. Developments and applications, *Russian Math. Surveys* **49**:6 (1994), 157-196.

[61] H. Schirmer, Fixed points, antipodal points and coincidences of n-acyclic valued multifunctions, *in* "Topological Methods in Nonlinear Functional Analysis", Contemp. Math., **21**, pp. 207-212, Amer. Math. Soc., Providence, RI, 1983.

[62] V. P. Soltan, "Introduction to the Axiomatic Theory of Convexity", Shtiinca, Kishinev, 1984 (Russian).

[63] N. E. Steenrod, The work and influence of Professor S. Lefschetz in algebraic topology, *in*: "Algebraic geometry and topology. A symposium in honor of S. Lefschetz", pp. 24–43, Princeton University Press, Princeton, N. J., 1957.

[64] R. M. Stephenson, Jr., Initially κ-compact and related spaces, *in* "Handbook of Set-Theoretic Topology" (K. Kunen and J. E. Vaughan, Eds.), pp. 603-632, Elsevier Science Publishers B.V., 1984.

[65] W. Takahashi, A convexity in metric spaces and non-expansive mappings I, *Kodai Math. Sem. Rep.* **22** (1970), 142-147.

[66] E. Tarafdar, Fixed point theorems in locally H-convex uniform spaces, *Nonlin. Anal., Theory, Meth., Appl.* **29** (1997) 9, 971-978.

[67] M. L. J. Van de Vel, "Theory of Convex Structures", North-Holland, 1993.

[68] ———, A selection theorem for topological convex structures, *Trans. Amer. Math. Soc.*, **336** (2), (1993), 463-496.

[69] J. W. Vick, "Homology Theory, An Introduction to Algebraic Topology", Academic Press, New York, 1973.

[70] A. Wieczorek, Compressibility: a property of topological spaces related to abstract convexity, *J. Math. Anal. Appl.* **161** (1991), 9-19.

[71] A. Wieczorek, The Kakutani property and the fixed point property of topological spaces with abstract convexity, *J. Math. Anal. Appl.* **168** (1992), 483-499.

[72] E. Zeidler, "Nonlinear Analysis and its Applications", Vol. I, Springer-Verlag, New York/Berlin, 1986-1991.

Peter Saveliev was born in Spassk Dalny, Russia on February 1965. He graduated from Moscow State University with a Diploma in Mathematics in 1987. Then he worked at the Institute for Precision Instruments, Moscow, Russia. He entered the Graduate College of the University of Illinois at Urbana-Champaign in 1994. His mathematical interests lie in Topology, Fixed Point Theory and its applications.